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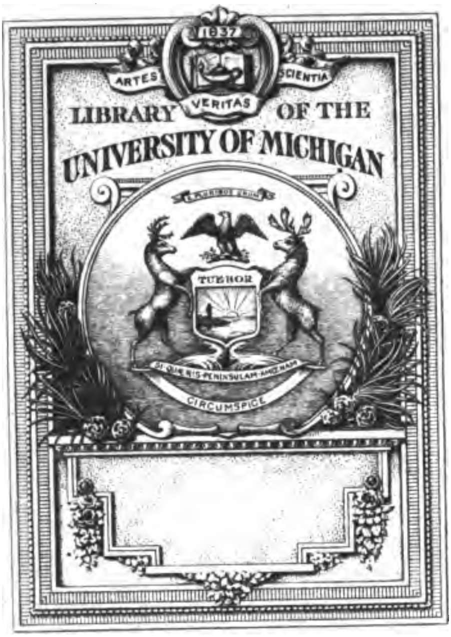
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AN

INTRODUCTION

TO

SOLID GEOMETRY,

AND TO THE STUDY OF

Chrystallography,

CONTAINING AN

INVESTIGATION OF SOME OF THE PROPERTIES

BELONGING TO

THE PLATONIC BODIES

INDEPENDENT OF THE SPHERE.

BY

*N.*athaniel *J. LARKIN, M. G. S,*

TEACHER OF CRYSTALLOGRAPHY AND MATHEMATICS.

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TO
G. B. GREENOUGH, Esq.
PRESIDENT,
THE VICE PRESIDENTS,
and
OTHER OFFICERS
of the
GEOLOGICAL SOCIETY,
THIS WORK
IS RESPECTFULLY DEDICATED,
AS AN ACKNOWLEDGEMENT OF THE FAVOURS RECEIVED FROM THEM,
AND THE SOCIETY IN GENERAL,
BY THEIR
FAITHFUL AND DEVOTED SERVANT,
THE AUTHOR.

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PREFACE.

CRYSTALLOGRAPHY, a science of modern discovery, has opened a wide field for the contemplation of those who have a taste for solid geometry, and whose investigations have been hitherto mostly confined to the five Platonic bodies, together with prisms and pyramids. These few solids, however, occupied much of the attention of the ancient mathematicians, especially of Plato, who first treated of them. Euclid, his cotemporary, is said to have composed his elements of geometry, for the express purpose of demonstrating such of the properties belonging to these solids with which he was acquainted. These properties are chiefly confined to the relations which the above mentioned solids have to the circumscribed sphere, or to each other when inscribed in the same sphere. Since his time various attempts have been made to extend the subject. The most successful appears to be that of Abraham Sharp, who, about the beginning of the last century, published a work, which is certainly ingenious; but not being furnished with the clue afforded by the study of Crystallography, and his attention being engrossed by his favourite object of pursuit, namely, tracing the relations between the angular solids with their inscribed or circumscribed spheres, he has distorted two of the finest and most interesting solids, the hexacontahedron and trapezohedron, for the sake of getting the greatest possible number of their angular extremities to touch the circumscribed sphere. This remarkable and beautiful property of the Platonic bodies has in general so facinated mathematicians that they have passed by the various other properties of these bodies, and their mutual relations independent of the sphere, as a barren field for investigation. That the reverse, however, is the case will, it is hoped, be conceded by all who attentively peruse the following pages. How far the author has done justice to the subject, must be left to the judgment of the reader to determine.

The arrangement of the work has been adopted for the sake of demonstrating the existence of the artificial series. It commences with a description of the Platonic bodies, and of those solids which naturally arise from them; it afterwards proceeds to treat of those angular solids which rank next in importance, and concludes with the solids comprehended under the the general terms Prisms, Pyramids, and Parallelopipeds; to which is subjoined a short description of the Cone, Sphere, and Cylinder.

In describing the three first of the Platonic bodies, namely, the tetrahedron, the cube, the octahedron, as well as the rhomboidal dodecahedron and trapezohedron, which are derived from the first three, he has shewn how any one of these five solids may be cut out of the remainder, in a very simple manner; which forms by no means the least interesting part of the following pages.

The demonstrations are placed at the end for the convenience of the reader. These commence with the sections of the axes of the natural solids, by planes passing through their various angular extremities at right angles to the axes: then follow the Theorems demonstrating the ratios subsisting between the volumes of the natural solids, when contained in each other, or in the same solid; concluding with some theorems demonstrating certain properties belonging to the Platonic bodies and their derivatives.

The demonstrations of the ratios subsisting between the volumes of the solids composing the artificial series, together with the various inclinations of their faces, are given in the appendix, which was drawn up by his friend Dr. Roget, who kindly undertook the laborious task of investigating and of calculating them. Some of the results of his investigations have also been given in the body of the work under the articles dodecahedron, icosahedron, triacontahedron, and hexacontahedron.

The most important subject in the following work to which the author would particularly direct the attention of the reader is, the existence of the double series of solids, which he has termed the natural and the artificial for the purpose of distinguishing them, and the various properties belonging to them. Among these are the following remarkable ones, namely, that the last solid in each series contains all the foregoing of the same series; the angular extremities of the latter being easily traced out in the surface of the former: and that the volumes of the contained solids in the last of the natural series, which is the trapezohedron, are commensurable with each other, and likewise with that of the trapezohedron which envelopes them. Another circumstance, perhaps not less surprising, is the simplicity of the numbers expressing their ratios, which are all rational; and in no instance, among the simple natural solids, have a greater factor than three. The Cuboctahedron, which is not a simple solid, but is only part of one, has constantly, in one of the numbers expressing the ratios between it and any one of the simple solids, a five for its greatest factor, the other term never has a greater factor than three, and in several instances it is less. The last of the novel part of the work is the description of seven distinct dodecahedrons, together with their remarkable connection with the cube.

There is considerable difficulty in defining the axis of a solid. The definition given in this work applies to all the simple solids, that is, solids formed of equal and similar faces, together with pyramids formed of equal and similar sides: but it does not include all pyramids, nor any of the prisms or cylinders. Since printing the above definition, a new one has been given by Professor Jameson of Edinburgh, in his treatise on mineralogy, communicated to him by Professor Mohs, who occupies the chair of the late celebrated Professor of Freyberg. On this definition Mohs has raised a new system of Crystallography, a short description of which system has been given in the above work with the definition. His description, however, has evidently not done justice to the subject; but a work expressly on the above system is announced for publication, in which no doubt ample compensation will be made. The definition itself has been much obscured in consequence of making use of the term figure to signify both a plane and a solid. Mohs's definition of the axis of a solid as given in the above mentioned treatise, is this: "When the section of a simple figure (solid) as a rhomboid or cube, affords, by means of a plane which does not pass through its centre, a regular, or equiangular, or equilateral figure, or one in which such a figure can be inscribed, the straight line which stands perpendicular on the middle point of the figure (face of the section) and passes through the centre of the figure (solid) is an axis."

This definition has given rise to a variety of names for different kinds of axes; for instance, when the section is an equilateral triangle it is called a rhomboidal axis, when it is a square it is called a pyramidal axis, and when it is a rhomb it is called a prismatic axis. Reasons are stated for so naming them, but some of them are unintelligible unless further explanations be given.

The idea, however, of classing the different kinds of axes may certainly be useful to solid geometry, for the purpose of defining the axes of various solids. In such case the first class may consist of those simple solids, which have opposite faces, that are parallel to each other; both the extremities of the axes of these solids are clearly marked out in the surface of the solids each terminating in an angular extremity. The second class may be called the *pyramidal axis,* or the axes of those solids which have an angular extremity opposite to each face; one extremity only of those axes terminates in an angular extremity of the solid, the other terminating in the centre of the opposite face. The third class may be called a *prismatic axis;* which is a line drawn from the centre of one base to the centre of the opposite base, neither of the extremities of these axes terminate in any of the angular extremities of the solid.

Trapezohedron is a term lately introduced by M. Brochant, to designate a solid, bounded by twenty four equal and similar trapezoidal faces, which solid had before no specific name. This term is equivocal, being equally applicable to two other solids, (see page 57), but these two solids have been little known, nor are they likely to present themselves to notice so frequently, as the one in question, on which account it may be allowed this preference. The term is very expressive of the quality of the faces, though it does not express the number of them, in which respect it differs from the names of the Platonic bodies. *Tetrahexahedron* meaning four times six sided would have been more analogous to them, though perhaps upon the whole not less exceptionable.

Rhombohedron has been introduced by the same Mineralogist instead of the term rhomboid; the first term means rhomb-sided and the last rhomb-like. The former does not express the number of the sides, but the quality of them. In order to have expressed the number of the sides it should have been named hexahedron, in which case the two varieties must have been distinguished by the terms acute hexahedron and obtuse hexahedron; but these are not expressive of its most remarkable property, namely, its being bounded by rhombs, and they would be less likely to be generally understood from their total want of similarity to the names by which these solids have hitherto been distinguished. The Rhombohedron was first known by the name *solid Rhombus;* but at that time no other variety was noticed than the acute rhombohedron, *fig.* 99, whose faces were each formed of two equilateral triangles. Some years ago Haüy introduced the term Rhomboid to express every variety, both acute and obtuse. This term rhomboid was formerly used to denote a parallelogram, whose sides are not all equal nor its angles right angles, which parallelogram was then named an *oblique angled parallelogram;* should the term rhombohedron be generally adopted, the above parallelogram might, with great propriety and convenience, have its original name restored.

Triacontahedron, or thirty-sided is a term introduced by Haüy.

Hexacontahedron, or sixty-sided, is introduced on account of its analogy with the last and the other terms.

Cubooctahedron derives its name from being intermediary between the cube and the octahedron, and because it is bounded by six squares, the number of faces of a cube, and eight equilateral triangles, the number of faces in an octahedron.

Excube derives its name from being formed from a cube, by having its angular extremities slightly truncated.

Exoctahedron, derives its name from being formed from an octahedron, by having its angular extremities slightly truncated. The preposition *ex* has been used as a prefix, on account of its being the most concise word that could be found to answer that purpose; excube literally meaning from a cube, and exoctahedron from an octahedron. The terms excube, cuboctahedron, and exoctahedron are expressive of three different intermediary stages, between the cube and octahedron, each of these stages or steps, with every intermediary one that can possibly occur, may be distinguished by the general appellation of cubo-octahedron a term introduced by Haüy for that purpose.

Icosadodecahedron derives its name, from being intermediary between the icosahedron and dodecahedron, and because it is bounded by twenty equilateral triangles, the number of faces in an icosahedron, and twelve pentagons, the number of faces in a dodecahedron.

Exdodecahedron derives its name from being formed from a dodecahedron, by having its angular extremities slightly truncated.

Exicosahedron derives its name from being formed from an icosahedron, by having its angular extremities slightly truncated.

Triexoctahedron is a term that has been introduced by Cowley, the solid designated by it is generated by the mutual intersection of a rhomboidal dodecahedron, an octahedron and a cube; it has been omitted in the body of the work by accident.

The term triangular dodecahedron among the definitions, has been introduced by mistake, instead of tetradodecahedron, the term triangular dodecahedron is general, and equally applicable to the scalene and bipyramidal dodecahedrons.

N. J. LARKIN.

Gee Street, Somers Town,
Feb. 2, 1820.

Sets of Solids to accompany this work may be had of Mr. Mawe, 149, Strand; Mr. Allen, near the College, *Edinburgh;* Mr. Clark, Sackville-Street, *Dublin;* Messrs. Harris & Co. *Hamburgh*, and of all Mathematical Instrument Makers in town and country.

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Natural Series.	*Artificial Series.*
1 Tetrahedron	6 Pentahedron
2 Hexahedron, or Cube	7 Dodecahedron
3 Octahedron	8 Icosahedron
4 Rhomboidal Dodecahedron	9 Triacontahedron
5 Trapezohedron	10 Hexacontahedron
11 Acute Rhombohedron	19 Triangular Pyramid
12 Obtuse Rhombohedron	20 Quadrangular Pyramid
13 Scalene Dodecahedron	21 Hexagonal Prism
14 Bipyramidal Dodecahedron	22 Hexagonal Pyramid
15 Tetradodecahedron	
16 Pentagonal Dodecahedron	23 Cone
17 Trapezoidal Dodecahedron	24 Sphere
18 Triangular Prism	25 Cylinder

◆

A LIST OF THE DERIVATIVES AND INTERSECTING SOLIDS, &c.

Intersecting Solids.	*Derivatives.*
26 Pentahedron	32 Excube
27 Re-octahedron	33 Cuboctahedron
28 Re-dodecahedron	34 Exoctahedron
29 Two Pentahedrons intersecting each other	
30 A Cube and an Octahedron intersecting each other	35 Exdodecahedron
31 A Rhomboidal Dodecahedron and Cuboctahedron intersecting each other	36 Icosadodecahedron
	37 Exicosahedron
	38 Triexoctahedron

VARIETIES OF SIMPLE SOLIDS.

39 Obtuse Octahedron with square base	43 Rhomboidal Octahedron
40 Acute Octahedron with square base	44 Pentagonal Dodecahedron
41 Rhombic Octahedron	45 Ditto
42 Rectangular Octahedron	

PARALELLOPIPEDS, OR QUADRANGULAR PRISMS.

Right Prisms	*Oblique Prisms*
46 with square base	50 with a square base
47 with a rhombic base	51 with a rhombic base
48 with a rectangular base	52 with a rectangular base
49 with a rhomboidal base	53 with a rhomboidal base

Private Lectures given, and Schools attended by the Author.

The author is requested to inform the reader, that a Mineralogical Dictionary is in the press, and will be published in the course of the spring. It will contain an explanation of the various terms used in Crystallography and a complete explanation of Haüy's Theory, illustrated by numerous engravings by Mr. and Miss Lowry, from original Drawings made expressly for the work.

CONTENTS.

Page

DEFINITIONS 1

TETRAHEDRON 5

The number of its faces, &c. *ib.*

Inclinations of its faces, edges and axes *ib.*

To extract the contained octahedron *ib.*

Square sections *ib.*

To extract the inscribed cube 6

To extract the inscribed rhomboidal dodecahedron *ib.*

To extract the inscribed trapezohedron *ib.*

The inclination of lines drawn from the angular extremities to the centre 8

Ratios subsisting between the altitude of the contained solids and some remarkable line in or upon the face of the tetrahedron *ib.*

Ratios subsisting between the volumes of the contained solids and the enveloping tetrahedron *ib.*

HEXAHEDRON or CUBE 9

The number of its faces, &c. *ib.*

Inclinations of its faces, edges and axes. *ib.*

The diagonals, &c. 10

Hexagonal sections *ib.*

To extract the contained tetrahedron from a cube 11

To extract the inscribed octahedron from a cube *ib.*

To extract the inscribed rhomboidal dodecahedron from a cube *ib.*

To extract the inscribed trapezohedron from a cube 12

Cuboctahedron 12

The inclination of lines drawn from the extremities of a cube to its centre *ib.*

Page
Ratios subsisting between the altitude of the contained solids with some remarkable lines in, or upon the face of, the enveloping cube .... ib.
Ratios subsisting between the volumes of the contained solids and the enveloping cube 13
OCTAHEDRON 14
Number of its faces, &c. .... ib.
Inclination of its faces, edges and axes .... ib.
Remarkable properties subsisting between the octahedron and the cube .... ib.
Varieties of octahedrons 17
To extract the inscribed tetrahedron from an octahedron .... ib.
To extract the inscribed cube from an octahedron 18
To extract the inscribed rhomboidal dodecahedron from an octahedron ib.
To extract the inscribed trapezohedron from an octahedron .... ib.
Ratios subsisting between the altitudes of the contained solids and some remarkable line in, or upon the face of, the enveloping octahedron 19
Ratios subsisting between the volumes of the inscribed solids and the enveloping octahedron .... ib.
DODECAHEDRON 20
Number of faces, &c .... ib.
Inclination of its faces, edges, and axes .... ib.
Various sections 21
Contained solids 22
ISOCAHEDRON 23
Number of faces, &c .... ib.
Inclinations of its faces, edges, and axes .... ib.
Various sections 24
PLATONIC BODIES 25
Parallel properties 27
RHOMBOIDAL DODECAHEDRON 29
Number of faces, &c. .... ib.
Inclination of the faces, edges, &c. .... ib.
Relations of five remarkable lines belonging to it 30
Remarkable properties of the solids contained in the rhomboidal dodecahedron 31
Various sections 32

Page.
To extract the contained tetrahedron from a rhomboidal dodecahedron 33
To extract the contained cube from a rhomboidal dodecahedron.... 34
To extract the contained octahedron from a rhomboidal dodecahedron.......... 34
To extract the contained trapezohedron from a rhomboidal dodecahedron .... ib.
Ratios subsisting between the altitudes of the contained solids, with some remarkable line in, or upon the enveloping rhomboidal dodecahedron.......... ib.
Ratios subsisting between the volumes of the contained solids and the enveloping rhomboidal dodecahedron.......... ib.
TRAPEZOHEDRON 35
Number of its faces, &c. .... ib.
Inclination of its faces, edges and axes 36
Remarkable properties of the solids contained in a trapezohedron .. 37
Ratios of the solids contained in a trapezohedron 38
Relation of the contained rhomboidal dodecahedron with the contained cuboctahedron 39
Hexagonal sections of a trapezohedron and its contained solids 42
Square sections of the contained solids 43
To extract the contained tetrahedron from a trapezohedron 44
To extract the contained cube from a trapezohedron .... ib.
To extract the contained octahedron .... ib.
To make a model to represent the contained rhomboidal dodecahedron and cuboctahedron intersecting each other.......... ib.
Sections of the longest axis of a trapezohedron 45
Sections of the shortest axis of a trapezohedron.......... ib.
Sections of the middle axis of a trapezohedron.......... 46
Remarkable relations between the trapezohedron and its contained rhomboidal dodecahedron .... ib.
Ratios subsisting between the altitude of the contained solids with some remarkable line in or upon the enveloping solid.......... 47
Ratios subsisting between the volumes of the contained solids and the enveloping trapezohedron.......... ib.
A remarkable solid to be derived from the trapezohedron 48
TRIACONTAHEDRON.......... 49
Number of faces, &c.......... ib.
Inclination of its faces, edges and axes .... ib.

Page

Properties of solids contained in a triacontahedron 50
Sections of the triacontahedron 51
HEXACONTAHEDRON 53
Number of faces, &c. .... *ib.*
Inclination of its faces, edges, and axes .... *ib.*
Various relations 54
Remarkable properties of the solids contained in a Hexacontahedron 55
Sections of the hexacontahedron 56
General observations 57
PENTAHEDRON 60
To extract a dodecahedron from a pentahedron 62
Remarkable properties 63
ARTIFICIAL SERIES 65
General observations .... *ib.*
Of an indefinite number of series of solids governed by one general law 67
RHOMBOHEDRON OR RHOMBOID 68
Acute .... *ib.*
Obtuse 69
SCALENE DODECAHEDRON 71
BIPYRAMIDAL DODECAHEDRON 72
TETRADODECAHEDRON 73
TRAPEZOIDAL DODECAHEDRON 74
PENTAGONAL DODECAHEDRON 75
Seven distinct dodecahedrons 76
PRISM 77
PYRAMID 78
PARALLELOPIPED 79
CONE 80
SPHERE .... *ib.*
CYLINDER .... *ib.*
DEMONSTRATIONS of a variety of properties belonging to the natural series contained in fifty-three Theorems 81
DEFINITIONS of the position of the different solids when contained in each other 83
Two Tables shewing the ratios of the volumes of the natural solids, ... 122
APPENDIX BY DR. ROGET 131

AN
INTRODUCTION
TO
SOLID GEOMETRY, &c.

DEFINITIONS.

NATURAL SERIES.

TETRAHEDRON, a solid, having four equilateral triangular faces, *fig.* 1.

HEXAHEDRON OR *CUBE*, a solid, having six square faces, *fig.* 2.

OCTAHEDRON, a solid, having eight equilateral triangular faces, *fig.* 3.

RHOMBOIDAL DODECAHEDRON, a solid, having twelve equal and similar rhombic faces, *fig.* 4.

TRAPEZOHEDRON, a solid, having twenty-four equal and similar trapezoidal faces; the short diagonals of its faces coinciding with the edges of a contained rhomboidal dodecahedron,* *fig.* 5.

Compound solids belonging to the natural series.

Excube, a solid having six regular octagonal and eight equilateral triangular faces, *fig.* 14,

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* There is a difficulty in defining this solid on account of the indefinite variety of solids capable of being constructed, having twenty-four equal and similar trapezoidal faces.

Cuboctahedron, a solid, having six square faces and eight equilateral triangular faces, *fig*. 15.

Exoctahedron, a solid, having six square faces and eight regular hexagonal faces, *fig*. 16.

Triexoctahedron, a solid, having eight equilateral triangular faces and eighteen square faces, *fig*. 16.

THE ARTIFICIAL SERIES.

PENTAHEDRON, a pentagonal pyramid, whose base is a regular pentagon, and whose sides are isosceles triangles, each having the angles at the base double the third angle, *fig*. 7.

DODECAHEDRON, a solid, having twelve regular pentagonal faces, *fig*. 8.

ICOSAHEDRON, a solid, having twenty equilateral triangular faces, *fig*. 9.

TRIACONTAHEDRON, a solid, having thirty equal and similar rhombic faces, *fig*. 10.

HEXACONTAHEDRON, a solid, having sixty equal and similar trapezoidal faces; its long diagonals coinciding with the edges of a contained triacontahedron.* *fig*. 11.

Compound solids belonging to the artificial series.

Exdodecahedron, a solid, having twelve regular decagons, and twenty equilateral triangles for its faces, *fig*. 20.

Icosadodecahedron, a solid, having twelve regular pentagonal faces, and twenty equilateral triangular faces, *fig*. 21.

Exicosahedron, a solid, having twelve regular pentagonal faces, and twenty regular hexagonal faces, *fig*. 22.

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* There is a similar difficulty in defining this solid as in the case of the trapezohedron; on account of the variety of solids capable of being constructed sixt,iav bugy equal and similar trapezoidal faces.

RHOMBOHEDRON or RHOMBOID, a solid bounded by six rhombic faces.

SCALENE DODECAHEDRON, a solid, bounded by twelve equal and similar scalene triangular face, *fig* 81.

BIPYRAMIDAL DODECAHEDRON, a solid, bounded by twelve equal and similar isosceles triangles, forming two hexagonal pyramids joined base to base, *fig.* 83.

TRIANGULAR DODECAHEDRON, a solid, bounded by twelve isosceles triangles, forming four triangular pyramids one on each face of a tetrahedron, *fig.* 78.

TRAPEZOIDAL DODECAHEDRON, a solid, bounded by twelve equal and similar trapezoidal faces, grouped together in threes, one set on each face of a contained tetrahedron, *fig.* 79 and 80.

PENTAGONAL DODODECAHEDRON, a solid, bounded by twelve equal and similar pentagons, each of which have four of their sides either greater or less than the fifth side; the faces are grouped together in pairs, each pair being united by the fifth side, *fig.* 76 and 77.

A Prism is a solid, whose two ends are similar and equal polygons, and parallel to each other; and whose sides connecting those ends or bases are parallelograms.

A Pyramid is a solid, having a polygon for its base, and having its sides triangles, whose vertices all meet in the same point, which point is the vertex of the pyramid.

There may be an indefinite variety of prisms and pyramids, distinguished from each other by the number of sides that bound their respective bases; as triangular pyramids or prisms, quadrangular, &c.

A Parallelopiped is a quadrangular prism, whose opposite sides are all parallel.

A CONE is a solid, having a circular base, the remaining part of its superfices being one continued curved surface terminating in a point or summit; from which point or summit, straight lines may be drawn on the curved surface to any point in the extremity of the base.

A CYLINDER is a solid, having two equal circular faces or ends opposite and parallel, united by a curved surface, which surface is so formed that straight lines may be drawn on it from any point in the extremity of one base to the corresponding point, in the extremity of the other base.

A SPHERE is a solid, having one continued curved surface, every point of which is equally distant from a point in the middle of the solid, which is its centre.

An Axis is a line, passing through the centre of a solid; drawn from any angular point that is formed by the meeting of equal plane angles to the opposite angle or face.

A Diagonal is a line drawn on one of the faces, from one angular point to the opposite angular point.

The Altitude of a solid is a perpendicular, drawn from the centre of the face of a solid terminating in the opposite angle, or face produced if necessary.

THE TETRAHEDRON.

This solid has Four faces,
Four angular extremities,
Six edges,
Twelve plane angles, and
Four axes.

All its faces are equal, all its solid angles are equal, all its edges are equal, all its plane angles are equal, and all its axes are equal.

THE mutual inclination of its faces is 70° 31′ 43″½. The inclination of its edges on its faces is 54° 44′ 8″¼. The inclination of its faces to its axis is 19° 28′ 16″½. The inclination of its edges to its axis is 35° 15′ 51″¾. The inclination of the axes to each other is 109° 28′ 16″½, and its plane angles are 60°.

The tetrahedron may be considered as the most simple of the regular solids, on account of its having the least number of sides. In other respects, however, it is not the most simple.

It differs from every other solid, whose faces are all equal, by having a solid angle opposite to each face. It is the most simple and most remarkable of pyramids, having as many bases as it has planes, and on whichever of these it is placed, it is a similarly formed pyramid. It contains the least solidity under a given surface of any of the natural or artificial solids.

If the edge *A C* (*fig.* 25) of a tetrahedron be bisected in *c* and a plane pass through the point of bisection parallel to one of its faces *A B D*, this face of the section will be equal to the equilateral triangle *c s a*, and the part cut off will be a regular tetrahedron *a c s C*: and if the other angular extremities *A*, *B*, and *D*, be removed in the same manner, four similar tetrahedrons will be obtained, and the remaining part of the original tetrahedron, will be an octahedron, which octahedron will be equal to the four tetrahedrons cut off, and consequently the volume of the octahedron will be to the volume of the original tetrahedron, which may be called the enveloping tetrahedron, as 1 : 2

Were the plane to pass through *c s.* (*fig.* 25,) and cut the solid parallel to the edge *C D*, the face of the section will be the square *c b f s*, and the solid will be divided into two equal parts. This section may be made in three different directions, and it has the peculiar property of cutting every face of the solid at the same time, in an equal and similar manner. All sections parallel to this section will be rectangles.

If a line *c b* (*fig.* 31.) be drawn through the centre *n* of the face of a tetrahedron, parallel to one of the edges *A B*, and a plane pass through this line, cutting the solid parallel to the opposite edge *D C*, and the same be done in every possible direction, there will result six planes, one for each edge, which, by their mutual intersection, will form a cube in the middle of the tetrahedron, and they will cut the edges of the tetrahedron into three equal parts. The alternate angular extremities of the cube will terminate in the centres of the faces of the tetrahedron, and its diagonals will be equal to the distance of the centres *n e* from each other: the inscribed cube will be to its circumscribing tetrahedron as 1 : 9.

If a line *a b* (*fig.* 37) be drawn through the centre *n* parallel to the edge *B A*, and a plane pass through it, cutting the solid at right angles to its opposite edge *D C*, and the same be done in every possible direction, the result will be twelve planes, three for each face of the tetrahedron, which by their mutual intersections, will form a rhomboidal dodecahedron in the middle of the tetrahedron, its alternate obtuse angular extremities will terminate in the centres of the faces of the tetrahedron, and its volume will be the volume of the circumscribing tetrahedron as 2 : 9.

Hence if a line be drawn through the centre of the face of a tetrahedron parallel to one of its edges, and a plane pass through it, cutting the solid at right angles to the edge opposite, it will coincide with the face of the inscribed rhomboidal dodecahedron. But if it cut the solid parallel to the last-mentioned edge, it will coincide with the face of the inscribed cube.

If a line *g f*, (*fig.* 43) be drawn through the centre *s* of the face *A B C*, of a tetrahedron parallel to one of the edges *B C*, and a plane pass through this line cutting the solid at right angles to the face *B D C*, adjoining that edge, and this be done in every possible direction, there will result twelve planes, two for each edge, or three for each face: these planes will cut off *B g* and *C f* equal to $\frac{1}{3}$, and *B d* and *c e* equal to $\frac{1}{6}$ of the edge of the tetrahedron, and will by their mutual intersection, form one half of the faces of the inscribed trapezohedron, which faces bound that set of its alternate obtuse angular extremities, which terminate in the centre of the faces of the circumscribed tetrahedron. The remaining set of obtuse angular extremities of the inscribed trapezohedron, will each be opposed to one of the angular extremities, of the circumscribed tetrahedron; and the faces of the trapezohedron bounding these last-mentioned obtuse angular extremities, may be formed by planes intercepting the angular extremities of the tetrahedron, in the following manner: make *A t a* equal to $\frac{1}{3}$ *A C*, and *C b* and *C c* equal to $\frac{4}{9}$ of the edge of the tetrahedron, if a plane pass through these three points, it will be at right angles to the face *B D C*, and will intercept one of the faces of the inscribed trapezohedron. If this be done in every possible direction, there will result twelve planes, three for each angular extremity of the tetrahedron, these twelve with the twelve first mentioned, will make together twenty-four; which by their mutual intersection will form a trapezohedron in the middle of the tetrahedron, whose alternate obtuse angular extremities will terminate in the centres of the faces of the circumscribed tetrahedron, their solidities will be to each other as 8 : 27.

Lines drawn from the angular extremities of a tetrahedron to its centre, will meet at angles equal to the obtuse plane angles of a rhomboidal dodecahedron, dividing the tetrahedron into four triangular pyramids.

The altitude of the contained cube is to the perpendicular *a e* (*fig.* 30) of the tetrahedron or the axis of the contained octahedron as	1 : 3
The altitude of the contained octahedron is to the altitude of the tetrahedron as	1 : 2
The altitude of the inscribed rhomboidal dodecahedron is to the edge of the tetrahedron as	1 : 3
The altitude of the inscribed trapezohedron is to the perpendicular on the face of the tetrahedron as 4 : 9 or	$2^2 : 3^2$
The volume of the inscribed cube is to the volume of the tetrahedron as 1 : 9, or	$1 : 3^2$
The volume of the contained octahedron is to the volume of the tetrahedron as	1 : 2
The volume of the inscribed rhomboidal dodecahedron is to the volume of the tetrahedron as 2 : 9 or .	$2 : 3^2$
The volume of the inscribed trapezohedron is to the volume of the tetrahedron as 8 : 27 or	$2^3 : 3^3$

If a cube, an octahedron, a rhomboidal dodecahedron or a trapezohedron, have a tetrahedron contained or inscribed in each of them, and another tetrahedron enveloping or circumscribing either of them, the contained tetrahedron is to the enveloping tetrahedron in all the four cases as 1 : 27 or . . . . $1 : 3^3$

THE HEXAHEDRON OR CUBE.

This solid has Six faces,
Eight angular extremities,
Twelve edges,
Twenty-four plane angles, and
Four axes.

All its faces are equal, all its solid angles are equal, all its edges are equal, all its plane angles are equal, and all its axes are equal.

The inclination of its faces to each other, the inclination of the edges to each other, and the inclination of the edges on the faces are all equal, being all right angles; this equality is not to be met with in any other solid.

The inclination of its faces to its axis is 30° 15′ 51″^{3}. The inclination of its edges to its axis is 54° 44′ 8″¼. The inclinations the axes to each other are 109° 28′ 16″½ and 70° 31′ 43″½.

It is the most simple of parallelopipeds, having its three dimensions, length, breadth and altitude equal. It is the most remarkable of prisms as the tetrahedron is of pyramids, having as many bases as it has faces, and on whichever face it is placed it is a similarly formed prism, and on account of the ease with which it can be manufactured, it is highly probable that it was the first solid ever made by art.

The diagonal of the cube coincides with the edges of two contained tetrahedrons, whose angular extremities mutually protrude through each others faces, (see *fig.* 26) the parts of the tetrahedron common to both is an octahedron whose angular extremites terminate in the centre of the faces of the cube; the angular extremities of each tetrahedron, coinciding with the alternate angular extremities of the cube.

If a cube be held by two of its opposite angular extremities as *A B* (*fig.* 36,) between the thumb and finger, it will revolve with ease on those extremities, which extremities when held in that position may be called its summits, and the line uniting them is one of the axes of the cube: the remaining six may in that position be called the lateral extremities. On account of the uniformity of this solid, any of its extremities may become its summits. Upon inspecting the cube in this position, it will be found that three of its edges terminate in one of its summits, and three in the other: the upper three may be called its superior edges, and the lower three its inferior edges; and the six which unite the extremities of these, the lateral edges. A line drawn from one of the summits to an opposite angular extremity on the same face as *A F* (*fig.* 36) or *B E*, is an oblique diagonal, and a line drawn uniting two opposite lateral extremities as *E C*, or *G F*, is an horizontal diagonal.

If the lateral edges be bisected, (as in *fig.* 34) and lines be drawn uniting the points of bisection, they will all be in the same plane; which plane will divide the cube into two equal parts, and the face of the section will be a regular hexagon. This section similar to the one already described in the tetrahedron, cuts every face of the solid in an equal and similar manner. If lines be drawn uniting the points of bisection with the summits, they will coincide with the edges of a contained bipyramidal dodecahedron.

If all the edges of a cube be bisected, and lines be drawn uniting each of the adjacent points of bisection, these lines will be parallel to the diagonals of the cube, and will mark out the extremities of the contained cuboctahedron, *fig.* 92, whose edges will coincide with those lines, and will be each equal to one half of a diagonal of the face of the cube: their volumes will be to each other as 5 : 6.

If a plane pass through three diagonals of a cube as *E C*, *E D*, and *D C* (*fig.* 36,) it will cut off one third of the axis *A B*, and the face of the section will be an equilateral triangle, the volume of the triangular pyramid *C D E A*, is to the volume of the cube as 1 : 6.

If the diagonals *D B*, *D E* and *D H*, (*fig.* 26) be drawn, and their extremities be united by the diagonals *H B*, *B D*, and *D H*, these six diagonals will coincide with the edges of the contained tetrahedron, and if the alternate angular extremities *A*, *C*, *F* and *G*, be removed by four planes each passing through three diagonals, a tetrahedron will be developed, whose volume will be to the volume of the cube as 1 : 3.

If all the diagonals of a cube be drawn, and each of its angular extremities be removed by planes, each passing through three diagonals, there will be eight planes, one for each angular extremity of the cube, which by their mutual intersection will form an octahedron in the middle of the cube, as (*fig.* 32.) The angular extremities of the inscribed octahedron, will terminate in the centres of the faces of the circumscribed cube: the volume of the octahedron will be to that of the cube as 1 : 6.

If all the edges of the cube be bisected, and planes pass through the points of bisection, parallel to the faces of the cube, they will divide the cube into eight small cubes.

If two adjacent edges *A D*, *D F*, (*fig.* 38,) be bisected and a plane pass through the points of bisection, *b* and *c*, parallel to the edge *D C*, the face of the section will be a rectangle. If all the edges be bisected, and planes pass through the points of bisection in a similar manner to the above, there will be twelve planes formed, one for each edge, which planes by their mutual intersection will from a rhomboidal dodecahedron, whose acute angular extremities will terminate in the centres of the faces of the cube. These two solids will be to each other as 1 : 4.

If one of the edges, *D F* (*fig.* 44,) of the cube, be bisected in *b*, and lines be drawn from the point of bisection to two opposite angles *A* and *C*, and the diagonal *A C* be drawn, these three lines will be in the same plane; which plane, will intercept one of the faces of the contained trapezohedron. Each of the angular extremities of the cube may be intercepted by three such planes, making in all twenty-four, which by their mutual intersection, will form in the middle of the cube a trapezohedron, whose acute angular extremities will terminate in the centres of the faces of the cube. The relation of the two solids to each other will be as 1 : 3.

Lines drawn from the angular extremities of a cube terminating in its centre, will meet at angles, equal to the obtuse and acute plane angles, of the faces of a rhomboidal dodecahedron. These lines will divide the cube into six square pyramids, whose vertices will terminate in the centre of the cube. If these pyramids be inverted, and placed on the face of a cube of equal size with the one from which they were made, they will form a rhomboidal dodecahedron.

The altitude of the contained tetrahedron is to the axis of the cube, as	2 : 3
The altitude of the contained octahedron is to the axis of the cube as	1 : 3

The altitude of the contained rhomboidal dodecahedron, is to the diagonal of the cube as . . . $1:2$

The altitude of the contained trapezohedron is to the perpendicular *A c fig.* 36, on the diagonal *F G* as . . . . $2:3$

The volume of the contained tetrahedron is to the volume of the cube as . . . . $1:3$

The volume of the inscribed octahedron is to the volume of the cube as $1:6$, or . . . . $1:2\times3$

The volume of the inscribed rhomboidal dodecahedron is to the volume of the cube as $1:4$ or . $1:2^2$

The volume of the inscribed trapezohedron is to the volume of the cube, as . . . . $1:3$

If a cube be contained in a tetrahedron, an octahedron, a rhomboidal dodecahedron or a trapezohedron, and each of these solids have another cube enveloping them, the contained cube will be to the enveloping cube in relation to the intervening

Tetrahedron, as $1:27$, or . . . . $1:3^3$
Octahedron, as $1:27$, or . . . . $1:3^3$
Rhomboidal Dodecahedron, as $1:8$, or . . . $1:2^3$
Trapezohedron, as $1:8$, or . . . . $1:2^3$

THE OCTAHEDRON.

This solid has Eight faces,
Six angular extremities,
Twelve edges,
Twenty-four plain angles, and
Three axes.

The faces are all equal equilateral triangles; all the angular extremities are equal; all the edges are equal; all the plane angles are equal; and all the axes are equal.

The inclination of its faces to its axis is 35° 15′ 51″¾. The inclination of its edges on its axes is 45°. The inclination of the axes to each other is 90°. The inclination of the faces united by its edges is 109° 28′ 16″½. The inclination of the faces united by their angles is 70° 31′ 43″½. And the inclinations of the edges to each other are 60° and 90°.

The octahedron may be formed of two quadrangular pyramids joined base to base, each of which pyramid is formed of four equilateral triangles. It has the property of being divisible in two other directions, into two similar pyramids.

It is remarkable on account of a number of its properties, being either the same as in the cube, or the converse of those in the cube. The number of edges in an octahedron, being equal to the number of edges in a cube. The number of plane angles in each are equal. The inclination of the faces on their axes are equal. And the number of faces in the one is equal to the number of solid angles in the other.

If the edge of an octahedron be bisected, and a plane pass through the point of bisection parallel to one of its faces, it will divide the octahedron into two equal parts; and the face of the section will be a regular hexagon, but if the plane pass through the point of bisection at right angles to the axis farthest distant, it will pass through the edge that is bisected, and bisect the axis, making the face of the section a square. This is the converse of what takes place in the cube: for if one of the edges of a cube be bisected, and a plane pass through the point of bisection, and cut the solid at right angles to the farthest axis, it will bisect that axis, making the face of the section a regular hexagon: but if it pass through the point of bisection, and cut the cube parallel to one of its faces, the face of the section will be a square. Hence, if an edge of either of the solids be bisected; the face of a section made by a plane passing through the point of bisection in one of the solids parallel to one of its faces, is similar to the face of a section made by a plane passing through the point of bisection in the other solid, at right angles to the axis farthest distant from the point of bisection. In one case the face of the section is a square, and in the other a hexagon.

If the solid angles of one of these solids (either the cube or the octahedron) be intersected by planes passing through the centres of its faces at right angles to its axis, these planes will by their mutual intersection form the other solid, and the angular extremities of the inscribed solid will in either case terminate in the centres of the faces of the circumscribed solid.

This transformation of one solid into the other will be better understood by attending to the connection subsisting between the excube, the cuboctahedron and the exoctahedron. If the solid angles of a cube be removed, as in the excube, the faces of the cube from being squares become octagons, if the solid part of the cube continue to be removed, parallel to the triangular faces till they meet together, as in the cuboctahedron, the remaining part of the original faces of the cube again become squares, but in an inverted position with relation to its original faces; and should the removal of the solid part of the cube be carried on in the same direction, the squares will gradually get less, and the triangular faces will cut or intersect each other, and become hexagons; at first they are hexagons with unequal sides, and afterwards regular hexagons; at which period the solid becomes an exoctahedron, and if placed on one of its square faces, will be evidently an octahedron with its solid angles removed, or truncated. Should the solid be cut still deeper in a parallel direction to its hexagonal faces, they will assume a triangular form, increasing in the triangular form till they become perfect equilateral triangles, when the solid will be an octahedron. A similar process carried on with the octahedron, by removing the solid angles, will produce a cube.

If all the edges of a cube or an octahedron be bisected, and planes pass through the points of bisection, cutting off the adjacent solid angles at right angles to their respective axes, they will each be converted into a cuboctahedron. Lines drawn from the angular extremities of an octahedron to its centre will meet at right angles, and divide the solid into eight triangular pyramids, four of which, applied to the faces of a tetrahedron, having equal faces with the octahedron, will convert the tetrahedron into a cube.

The octahedron by elongating one of its axis, becomes elongated; but is still capable of being divided into two pyramids, whose common base will be a square; and if one of its axes be shortened. It is still divisible into two square pyramids; the first has an acute the last has an obtuse aspect, they are both distinguished by the title of an octahedron whose pyramids have a square base, *fig.* 75 *and* 76.

If three axes are changed, the octahedron is capable of being divided into two pyramids in three different directions, whose bases will be rhombs in each direction; whence it derives the title of an octahedron, whose pyramids have a rhombic base, *fig.* 77.

If the octahedron be so distorted that the common base of one set of pyramids is rectangular, it is called a rectangular octahedron (*fig.*78.) but should the common base of the pyramids, be an oblique 'angled parallelogram; it may be distinguished by the name of an octahedron whose pyramids has a rhomboidal base, *fig.* 79.

Each octahedron has its peculiar tetrahedron, by means of which the octahedron may be converted into a parallelopiped.

If from the six edges, which terminate in the angular extremities of the face *A C D*,. (*fig.* 27.) viz. *A E* and *A B*, *C B* and *C F*, *D F* and *D E* be taken, *A e*, *A f*, *C a*, *C b*, *D c* and *D d*, equal to one third of the edge of the octahedron, *e f a b c d* will be in the same plane, which plane will intersect a face of the contained tetrahedron. If similar planes intersect the solid, parallel to the alternate faces *A E B*, *B C F*, and *F E D*, they will by their mutual intersection, form a tetrahedron, in the middle of the octahedron, whose angular extremities will terminate in the centres of the alternate faces of the octahedron. The length of the edge of the inscribed tetrahedron will be equal to the distance of the centres of two adjoining faces of the octahedron. Its volume is to the volume of the octahedron as 2 : 27.

If from the four edges, terminating in the angular extremity *A*, (*fig.* 38) viz. *A B*, *A C*, *A D*, and *A E*, there be taken *A d*, *A b*, *A e* and *A n*, equal to two thirds of the edge of the octahedron, *d b e n*, will be in the same plane, which plane will intercept a face of the contained cube and pass through the centres of the faces of the octahedron: if each solid angle of the octahedron be intercepted in a similar manner there will be six planes, which, by their mutual intersection will form a cube in the middle of the octahedron, whose angular extremities will terminate in the centres of the faces of the octahedron. The inscribed cube is to the octahedron as 2 : 9.

If from the four edges terminating in the extremities of the edge *D C*, (*fig.* 39) viz. *D A*, *C A*, *C B* and *D B*, there be taken *D a*, *C b*, *C d* and *D e*, equal to one third of the edge of the octahedron; and *D n*, and *C c*, equal to $\frac{1}{2}$ *D E* and *C F*, *a b c d e n*, will be in the same plane, and will coincide with the face of the contained rhomboidal dodecahedron: if each edge of the octahedron be affected by a similar plane, there will be twelve planes, which, by their mutual intersection, will form a rhomboidal dodecahedron in the middle of the octahedron, whose obtuse angular extremities will terminate in the centres of the faces of the octahedron. The two solids will be to each other as 4 : 9.

If from *D A* and *D C*, (*fig.* 45) there be taken *D b* and *D a*, equal to two thirds the edge of the octahedron, and from *D E* and *D F*, there be taken *D e* and *D n* equal to $\frac{2}{3}$ *D E* and *D F*, *b a n c* will be in the same plane, which plane is perpendicular to *D F E*, and coincides with a face of the contained trapezohedron: the solid angle *D*, may be affected in a similar manner by four such planes; and if each solid angle be affected by four such planes, there will be altogether twenty-four planes, which, by their mutual intersection, will form a trapezohedron in the middle of the octahedron; and its obtuse angular extremities will terminate in the centres of the faces of the octahedron. They will be to each other as 16 : 27.

The altitude of the inscribed tetrahedron is to the altitude of the octahedron as . . . . 2 : 3

The altitude of the inscribed cube is to the axis of the octahedron as . . . . 1 : 3

The altitude of the inscribed rhomboidal dodecahedron is to the edge of the octahedron as . . . . 2 : 3

The altitude of the inscribed trapezohedron is to the perpendicular on the face of the octahedron as 8 : 9 or $2^3 : 3^2$

The volume of the inscribed tetrahedron is to the octahedron as 2 : 27 or . . . . $2 : 3^3$

The inscribed cube is to the octahedron as 2 : 9 or 2 : 3

The inscribed rhomboidal dodecahedron is to the octahedron as 4 : 9 as . . . . $2^2 : 3^2$

The volume of the inscribed trapezohedron is to the octahedron as 16 : 27 or . . . . $2^4 : 3^3$.

If an octahedron be contained in a tetrahedron, a cube, a rhomboidal dodecahedron, or a trapezohedron, and another octahedron be circumscribed about each; the circumscribed octahedron, will be to the contained octahedron, in the case of the intervening

Tetrahedron, as 1 : 27 or . . . . $1 : 3^3$
Octahedron, as 1 : 27 or . . . . $1 : 3^3$
Rhomboidal dodecahedron, as 8 : 27 or . . . . $2^3 : 3^3$
Trapezohedron, as 8 : 27 or . . . . $2^3 : 3^3$

THE DODECAHEDRON.

This solid has Twelve faces,
Twenty angular extremities,
Thirty edges,
Sixty plane angles and
Ten axes.

The faces are all equal regular pentagons : all its angular extremities are equal : all its angles are equal : and all its axes are equal.

The inclination of its faces is 116° 33′ 54″. The inclination of the edge on the face is 121° 43′ 3″. The inclination of the edge on the axis is 69°, 5′ 41″½. And the inclination of the face on the axis is 52° 37′ 21″½.

If the Platonic bodies be arranged according to the number of their faces, this will be the fourth agreeably to that order. The dodecahedron would appear to have derived its origin from an attempt to make a solid of regular pentagons after the discovery of the tetrahedron and cube; the one being formed of equilateral triangles, and the other of squares.

If a dodecahedron be held between the thumb and finger by two of its opposite angular extremities, it may revolve on them; and the line uniting them will be one of its axes; three of its faces will meet together at one extremity, and three will meet at the other extremity of each of its axis : these three faces may be called the superior or the inferior, according as one or other set is uppermost. The remaining six faces may be called its lateral faces; these may be distinguished, and one half may be called the inferior, and the other half the superior lateral faces.

If any two opposite edges in the same face, such as *i k* and *C k* (*fig.* 56) be bisected in *d* and *c*, and a plane pass through the points of bisection, cutting the solid at right angles to its farthest axis *E F*; it will divide the solid into two equal parts, and the face of the section, will be the regular hexagon *f a b c d e*. The dodecahedron having ten axes, there are consequently ten different directions in which this section may be made. There are likewise twenty other ways, in which sections may be made in this solid, having regular hexagonal faces.

If two adjoining edges *i n* and *i k* (*fig.* 56) be bisected in *g* and *d*, and a plane pass through the points of bisection, cutting the solid parallel to the face *A B C D E*, it will divide it into two equal parts, and the face of the section will be a regular decagon *g o a p d*.

If a diagonal *a b*, be drawn on the face *a c b d e*, (*fig.* 58,) and a plane pass through it, cutting the solid at right angles to the nearest axis *c i*, it will pass through the diagonals *a G* and *G b*, and the face of the section will be an equilateral triangle. If it pass through the diagonal *a b*, parallel to the edge *c G*, it will likewise pass through the diagonals *a F*, *F A* and *A b*, and the face of the section will be a square, which will be a face of a contained cube. If it pass through the diagonal *a b* at right angles to the axis *G P*, it will pass through *a E*, *E n*, *n l*, *l B* and *B b*, the face of the section will be a hexagon with unequal sides. If the plane pass through *a b*, and cut the solid parallel to the face *P q e d o*, the face of the section will be a regular pentagon.

Hence the faces of the section, or the interior planes of this solid will consist of equilateral triangles, squares, pentagons, hexagons, decagons, and rectangles, some of these planes will form a variety of solids by their intersection in the interior of the dodecahedron.

The pentagonal planes, by their mutual intersections, will form a small dodecahedron in the middle of the solid ; whose faces will be parallel to the faces of the exterior dodecahedron.

Six square planes may be taken, in such a manner, as not to intersect each other in the interior of the solid, but only in its surface ; these will form, by their mutual intersection, a cube, whose edges will be visible in the surface of the dodecahedron. If the square planes be taken in every possible direction, they will, by their mutual intersections, form five cubes, whose angular extremities will all terminate in the angular extremities of the dodecahedron ; as each cube has two tetrahedrons within it, therefore, the dodecahedron has ten tetrahedrons mutually intersecting each other, whose angular extremities also terminate in the angular extremities of the dodocahedron ; their faces will be in a direction parallel to the triangular sections. The cubes will form, by their mutual intersection, a triacontahedron in the middle of the dodecahedron.

Hence, if all the edges of a dodecahedron be bisected, and planes pass through the points of bisection in a direction parallel to the triangular sections ; they will by their mutual intersection, form five octahedrons, whose angular extremities will terminate in the points of bisection.

THE ICOSAHEDRON.

This solid has Twenty faces,
Twelve angular extremities,
Thirty edges,
Sixty plane angles, and
Twelve axes.

The faces are equal, the solid angles are equal, the edges are equal, the plane angles are equal, and the axes are equal.

The mutual inclination of its faces is 138° 11′ 23″. The inclination of the face on the edge is 110° 54′ 18″½. The inclination of an edge on the axis adjoining is 58° 16′ 57″, and the inclination of a face on the adjoining axis is 52° 37′ 21″½.

This solid is the last of the Platonic bodies ; and it is highly probable that it was the last discovered after it was known that more than one solid could be made equilateral triangles, by the discovery of the octahedron.

If an icosahedron be held between the thumb and finger by two of its opposite angular extremities it may revolve upon them, and the line uniting these extremities, will be one of its axes. Five of its faces will meet together at one extremity of the axis, and five will meet at the other extremity of the axis; these faces may be called the superior or inferior faces, according as one or the other is uppermost; and the remaining ten faces, which do not come in contact with the axis, may be called the lateral faces; five of which may be called the superior, and five the inferior.

The icosahedron and dodecahedron have a similar relation to each other that the cube and octahedron have. For the number of their edges are equal, the number of their plane angles are equal, and the inclination of their faces on their axes are equal; this is the case with the lateral faces, as well as with those which are in contact with the axis: and likewise he number of angular extremities in the one, is equal to the number of faces in the other.

If all the solid angles of a dodecahedron be truncated by planes at right angles to its axis, to such a degree that the pentagonal faces become regular decagons, an exdodecahedron (*fig*. 20) will be formed, of regular decagons and equilateral triangles. If this removal take place still deeper in the same direction, so that the extremities of the triangles touch each other, the solid will then assume the form of the icosadodecahedron (*fig*. 21), which is bounded by equilateral triangles and entagons. Should it be cut still deeper in the direction parallel to its triangular faces, the triangular faces will intersect each other, losing their angular extremities; they will then be hexagons with unequal sides, and afterwards perfect hexagons; when the solid will assume the form of the exicosohedron, (*fig*. 22). If this process be continued they will again become hexagons with unequal sides, but in the contrary direction to what they formerly were, and at last perfect triangles; when the pentagonal faces will disappear, and the solid will become an icosahedron. The solid angles of an icosahedron removed in a similar manner at right angles to its axis, would go through a similar process, till it became a dodecahedron.

The transition of the dodecahedron into the icosahedron is very similar to the transition of the cube into the octahedron; the principal difference is, that there are pentagons instead of squares, and decagons instead of octagons, for *fig.* 14 is bounded by octagons and triangles, and *fig.* 20 by decagons and triangles, *fig.* 15 is bounded by squares and triangles, and *fig.* 21 by pentagons and triangles, *fig.* 16 is bounded by hexagons and squares, and *fig.* 22 by hexagons and pentagons.

If two adjacent edges *uF* and *ud*, (*fig.* 57) be bisected in *a* and *b*, and a plane pass through the points of bisection, cutting the solid at right angles to the axis *A B*, farthest distant, it will divide the solid into two equal parts, and the face of the section will be a regular decagon *a b s v p*; but if it cut the solid at right angles to its nearest axis *u d*, the face of the section will be a regular pentagon *a b g n e*.

Planes passing through the centre of the faces of an icosahedron at right angles to its axis, form by their mutual intersection, a dodecahedron in the middle of the solid, whose angular extremities terminate in the centres of the face of the icosahedron.

If one of the edges of either of the solids be bisected, and through the points of bisection, a plane cut the solid at right angles to the axis, terminating in that edge, and should the remaining solid angles of both the icosahedron and dodecahedron be removed in a similar manner; each of the solids would be converted into an icosadodecahedron; (*fig.* 21,) having twelve regular pentagonal faces, and twenty equilateral triangular faces.

The above five solids are called the regular bodies, and likewise the Platonic bodies, because Plato was the first who treated of them; however, it is said that they were invented or constructed, above 200 years before his time by Pythagoras.

They are the only solids that can be made having equilateral and equiangular faces, without having at the same time re-entering angles.—*See the Demonstrations.*

If re-entering angles be admitted, another regular dodecahedron and octahedron may be formed, which will precisely answer the definitions in every other respect, and on account of their having re-entering angles, one may be called a re-dodecahedron and the other a re-octahedron to distinguish them from the rest. These solids are made by placing pyramids on each face of the ordinary dodecahedron and octahedron of such an altitude, that the sides of the pyramids, may be in the same plane with the faces of the solid, adjacent to that face on which the pyramid stands. Thus the re-octahedron is formed by placing a tetrahedron on each face of the octahedron, as (*fig.* 6,) and the re-dodecahedron (*fig.* 24) is formed by placing a pentagonal pyramid on each face of the dodecahedron.—*See the Demonstrations,*

Four of the Platonic bodies, viz. the cube, the octahedron, the dodecahedron and the icosahedron, are remarkable on account of the last two having similar properties subsisting between them, that the first two have. On account of the great importance of these properties, and the remarkable conclusion to be drawn from them, they will be recapitulated. In the first nine cases about to be enumerated, the same words being equally applicable, to the properties subsisting between the cube and octahedron, as subsists between the dodecahedron and icosahedron, they are not repeated. But the two last, on account of the necessity of introducing the names of the solids, were obliged to be repeated, and for the sake of comparison are placed in parallel columns; the properties themselves may be called parallel properties subsisting between the cube and octahedron on one side and the icosahedron and dodecahedron on the other.

1. The number of plane angles in one solid is equal to the number of plane angles in the other.

2. The number of edges in one solid is equal to the number of edges in the other.

3. The number of angular extremities in one solid is equal to the number of faces in the other.

4. If either of the solids have their angular extremities removed by planes, passing through the centre of its faces, at right angles to the adjoining axis, these planes, by their mutual intersection, will form the other solid.

5. If either of the solids have their edges removed by planes parallel to those edges passing through the centres of the adjoining faces, these planes, by their mutual intersection, will form a similar solid in both cases, whose faces will be equal and similar rhombs.

6. The inclination of the corresponding faces of either of the solids on their respective axes is equal.

7. If lines be drawn from the angular extremities of either of the solids to its centre, it will divide it into as many pyramids as the solid has faces; and if the two solids can be inscribed in the same sphere, or if their axes be equal, the altitudes of the two kinds of pyramids will likewise be equal.

8. If the two solids can be inscribed in the same sphere, or if their axes be equal, the faces of both of them may be inscribed in the same circle.

9. If the edges of either of the solids be bisected, and through the points of bisection planes cut the solid at right angles to their adjacent axis, they will each be converted, into

a solid, compounded of the two solids, having two sets of faces; one set of faces will be equal in number, and similar in shape to the faces of one of the solids; and the other set of faces will be equal in number, and similar in shape to the faces of the other solid.

10. If either of the solids have two adjacent edges bisected, and a plane pass through the points of bisection and cut the cube at right angles to its farthest axis, and the octahedron parallel to one of its faces, the face of the section will be the same in both cases.	10. If either of the solids have two adjacent edges bisected, and a plane pass through the points of bisection and cut the icosahedron at right angles to its farthest axis, and the dodecahedron parallel to one of its faces, the face of the section will be the same in both cases
11. If each face of the octahedron have a pyramid placed upon it, whose sides are in the same plane with the adjacent faces of the octahedron; their summits will be at equal distances, and lines drawn connecting these summits would form a cube.	11. If each face of a dodecahedron have a pyramid placed upon it, whose sides are in the same plane with the adjacent faces of the dodecahedron; their summits will be at equal distances, and lines drawn connecting these summits would form an icosahedron.

The above parallel properties of these four solids, shew that the dodecahedron and the icosahedron, not only form part of a continued series with the cube and octahedron under the title of Platonic bodies, but that they occupy a parallel position in a distinct series of solids, and from the two last comparisons, it would appear that the icosahedron answers to the cube, and the dodecahedron to the octahedron : but more will be said on this subject when the observations on the series, beginning with the tetrahedron are concluded.

From what has been said respecting the tetrahedron, the cube and the octahedron, it must appear that there is an intimate relation subsisting between them and the rhomboidal dodecahedron and trapezohedron; and from what will be said of the two last, it will be manifest that these five solids form a distinct series.

THE RHOMBOIDAL DODECAHEDRON.

This solid has Twelve faces,
Fourteen angular extremities,
Twenty four edges,
Forty eight plane angles, and
Seven axes.

Its faces are all equal and similar. It has two varieties of solid angles, six acute, and eight obtuse. All its edges are equal, it has two varieties of plane angles, twenty four acute, and twenty four obtuse. It has two variety of axes, three long and four short.

The mutual inclination of those faces united by their edges is 120°. The inclination of those faces united by their acute angles is 90°. The inclination of its faces with its edges terminating in the obtuse angles of the face, is 125° 15′ 51″$\frac{3}{4}$. The inclinations of those edges to each other, which terminate in the acute angular extremities, are 109° 28′ 16″½ and 70° 31′ 43″½. Its plane angles are 109° 28′ 16″½ and 70° 31′ 43″½.

The long diagonal is to the short as $\sqrt{2} : 1$. The short diagonal is to the edge as $2 : \sqrt{3}$ The longest axis, the shortest axis, the altitude which is the longest diagonal, and the shortest diagonal, are to each other as 2, $\sqrt{3}$, $\sqrt{2}$, 1 and $\frac{1}{2}\sqrt{3}$, or $\sqrt{4}$, $\sqrt{3}$, $\sqrt{2}$, $\sqrt{1}$, and $\sqrt{\frac{3}{4}}$.

The inclination of its faces with the shortest axis is 54° 44′ 8″¼. The inclination of its edges with the longest axis, is 54° 44′ 8″¼. The inclination of its faces with the longest axis, is 45°. The inclination of its edges with the shortest axis, is 70 °31′ 43″½.

The edges bounding the acute solid angles of a rhomboidal dodecahedron have this remarkable property, that the inclination of those adjoining each other, are the supplements of those that cross the solid angle; consequently, the inclination of the edges that cross the acute solid angles, is equal to the obtuse angles of the faces, and equal to the inclination of the edges of the obtuse solid angle. In the obtuse solid angle, the inclination of the edge on the face, is equal to half the acute plane angle and a right angle.

The rhomboidal dodecahedron, may be considered as generated, by the intersection of the edges of four hexagonal prisms, or the intersection of the sides of three square prisms. The axes of the hexagonal prisms will coincide with the shortest axes, and the axes of the square prisms, will coincide with the longest axes.

A rhomboidal dodecahedron may be formed of an octahedron, by applying eight triangular pyramids, such as has been described, (page 8) formed from a tetrahedron, whose faces are equal to the faces of the octahedron; consequently a rhomboidal dodecahedron is equal to an octahedron and two tetrahedrons, whose edges are each equal to the long diagonals of the rhomboidal dodecahedron.

The long diagonals of the faces concide with the edges of a contained octahedron; and its acute angular extremities, with the angular extremities of the octahedron.

The short diagonals of the faces concide with the edges of a contained cube; and its obtuse angular extremities coincide with the angular extremities of the cube.

The angular extremities of the contained octahedron and cube, mutually protrude through each others faces, as in fig 70, and their edges mutually bisect each other; the angular extremities of the octahedron, forming square pyramids; and the angular extremities of the cube forming triangular pyramids.

If the angular extremities of the octahedron be united to the angular extremities of the cube by lines, these lines will form the edges of an enveloping rhomboidal dodecahedron; and the edges of the octahedron and cube will be the diagonals of its faces. The volume of the octahedron is to that of the cube as 4 : 3. The protruded part of the octahedron is to the protruded part of the cube as 3 : 1; and one of the square pyramids is to one of the triangular pyramids as 4 : 1.

If an octahedron and a cube be taken of such proportion, that the edges of the octahedron be equal to the diagonals of the cube, or the edges of the cube be equal to half the axis of the octahedron; and the edges of the two solids be bisected, and lines be drawn on the faces of the solids, connecting the adjacent points of bisection, these lines will mark off the solid angles in each solid, forming the two kinds of pyramids above mentioned. The surface of the octahedron will be divided by these lines into triangles, so that its angular extremities will have the appearance of quadrangular pyraminds, with a triangular space on each face connecting them: and the extremities of the cube, will have the appearance of triangular pyramids, with a square space on each face connecting them. The bases of the pyramids of one solid will exactly fill the spaces marked out on the faces of the other; and if the pyramids be removed from one solid and placed on the spaces of the other, they will form an exact representation, of the cube and octahedron intersecting each other in a rhomboidal dodecahedron (*fig.* 70.)

If a plane pass through one of the long diagonals of a rhomboidal dodecahedron, cutting the solid at right angles to the face, it will divide it into two equal parts, and the face of the section will be a square; this section will cut the contained tetrahedron, cube, and octahedron, into two equal parts; and the faces of all the sections will be squares.

If through the point, formed by the intersection of the short and long diagonals of the face of a rhomboidal dodecahedron, a line be drawn perpendicular to one of the edges of that face; and through this line a plane cut the solid at right angles to the face, it will divide the solid into two equal parts; and the face of the section will be a regular hexagon: this section will also cut the contained cube and octahedron into two equal parts; and the faces of their sections will be regular hexagons: the face of the section of the two tetrahedrons in the cube will be a six pointed star, formed by the intersection of two equilateral triangles.

Sections may be made in six different directions, having regular octagonal faces.

A plane passing through one of the long diagonals at right angles to the longest axis, bisects that axis, but if it pass through the obtuse angular extremities, at right angles to the same axis, it will cut off one-fourth.

A plane, passing through the centre of the face of a rhomboidal dodecahedron, and cutting the solid at right angles to the most distant short axis, bisects that axis: but passing through an obtuse angular extremity, at right angles to the same axis, it cuts off one third, and passing through an acute angular extremity, at right angles to the same axis, it cuts off one sixth.

If three planes pass through one of the obtuse angular extremities of a rhomboidal dodecahedron, one at right angles to each of the edges terminating in that extremity; these three planes will, at the same time, pass through the three alternate angular extremities; if a fourth plane pass through those three alternate angular extremities, the four planes will by their mutual intersections form a tetrahedron, in the middle of the solid. Each of the planes will affect the rhomboidal dodecahedron in a similar manner as the plane *C e A g B d*, (*fig.* 28), which passes through the obtuse angular extremity *C* at right angles to *C D*, cutting one third from three of the edges of the rhomboidal dodecahedron, as *E g* which is equal to one third of *E F*. The volume of the rhomboidal dodecahedron is to its contained tetrahedron as 6 : 1.

If a plane pass through the short diagonal *G C*, (*fig.* 34), at right angles to the longest axis *H N*, it will likewise pass through three other short diogonals, *G A*, *A B*, and *B C*; if this take place in every possible direction through the short diagonals, these planes will, by their mutual intersections, form an octohedron in the middle of the solid: each of these planes will cut off one fourth from the axis to which they are at right angles. Their volumes will be to each other as 2 : 1.

If a plane pass through the long diagonals *A B fig.* 40, at right angles to the shortest axis *D E*, it will pass through three other long diagonals, *B C* and *C A*; if this be done in every possible direction, these planes will, by their mutual intersection, form an octahedron in the middle of the solid. The volumes of the two solids will be to each other as 2 : 3.

Hence the volumes of the rhomboidal dodecahedron, and of its contained octahedron, cube, and tetrahedron, will be to each other as the numbers 6, 4, 3, and 1.

If the two edges *F C* and *F B fig.* 46, be bisected in *d* and *e*, and a plane pass through the points of bisection parallel to *F A*, it will bisect *A H* and *A I*, and cut of *A c* equal to $\frac{3}{4}$ *A G*. If this be done in every possible direction, there will be a plane for every edge, making twenty-four in all, which by their mutual intersection, will form a trapezohedron in the middle of the solid. The intermediary angles of the contained trapezohedron, will be in the centre of the faces of the rhomboidal dodecahedron. The volume of one will be to the volume of the other as 9 : 16.

The altitude of the contained tetrahedron is to the edge of the rhomboidal dodecahedron as 4 : 3 or . . $2^2 : 3$

The altitude of the contained cube is to the longest axis as . . . . 1 : 2

The altitude of the contained octahedron is to the edge of the rhomboidal dodecahedron as 4 : 3 . . . $2^2 : 3$

The altitude of the inscribed trapezohedron is to the perpendicular on the face of the rhomboidal dodecahedron as . . . . , ... 3 : 2

The volume of the contained tetrahedron is to the rhomboidal dodecahedron as 6 : 1 or . . . . $2 \times 3 : 1$

The volume of the contained cube is to the rhomboidal dodecahedron as . . . . $1 : 2$

The volume of the contained octahedron is to the rhomboidal dodecahedron as . . . . $2 : 3$

The volume of the inscribed trapezohedron is to the rhomboidal dodecahedron as 9 : 16 or. . . . . . $3^2 : 2^4$

If a rhomboidal dodecahedron be contained in a tetrahedron a cube, an octahedron, or a tetrahedron, and another rhomboidal dodecahedron envelope each of these solids, the contained rhomboidal dodecahedron will be to the enveloping rhomboidal dodecahedron, in relation to the intervening

Tetrahedron as 1 : 27 or . . . . $1 : 3^3$
Cube as 1 : 8 or . . . . $1 : 2^3$
Octahedron as 8 : 27 or . . . . $2^3 : 3^3$
Trapezohedron as 27 : 64 or . . . . $3^3 : 2^6$

THE TRAPEZOHEDRON

Has Twenty-four faces,
Twenty-six angular extremities
Forty-eight edges,
Ninety-six plane angles, and
Thirteen axes.

Its faces are all equal and similar, it has three varieties of solid angles, eight obtuse, six acute, and twelve intermediary

It has two varieties of edges, twenty-four long and twenty-four short. It has three varieties of plane angles, twenty-four obtuse, twenty-four acute, and forty-eight acute nearly approaching right angles. It has three varieties of axes, three long, six middle, and four short.

The inclination of the faces united by the shorter edges is 146° 26′ 33″. The inclinations of the faces united by the longest edges is 131° 48′ 36″. The inclination of the faces united by the acute plane angles is 109° 28′ 16″ ½. The inclination of the faces united by the intermediary plane angles is 120°. The acute plane angles are 78° 27′ 46″. The obtuse plane angles are 117° 2′ 8″. The intermediary plane angles are 82° 15′ 3″. The inclination of the faces on the longest axis is 54° 41′ 8″¼. The inclination of the faces on the middle axis is 60°. The inclination of the faces on the shortest axis is 70° 31′ 43″½. The inclination of the longest edges crossing the acute solid angles is 126° 52′ 11″½. The inclination of the longest edges crossing the intermediary solid angles is 143° 7′ 48″½. The inclination of the shortest edges crossing the intermediary solid angles is 129° 31′ 16″½. The inclination of the longest edges on the longest axis is 63° : 26′ : 5″¾. — The inclination of the longest edges on the middle axis is 71° 33′ 54″¼. The inclination of the shortest edge on the middle axis, is 64° 45′ 38″¼. The inclination of the shortest edge on the shortest axis is 80° 38′ 30″.

The longest diagonals of the face are to the shortest as $4\sqrt{2} : 3\sqrt{3}$.

The longest edge is to the shortest edge as $2\sqrt{5} : 11$.

The short diagonal is divided into two unequal parts by the long diagonal, the longest part of which is to the shortest part, as 2 : 1. The long diagonal is bisected by the short: the half of the long diagonal will be to the greater part of the short as $\sqrt{2} : \sqrt{3}$; and to the lesser part as $2\sqrt{2} : \sqrt{3}$.

The longest axis, the middle axis, the shortest axis, and the altitude of the trapezohedron, are to each other as 6, $4\sqrt{2}$, $3\sqrt{3}$ and $2\sqrt{6}$.

If lines be drawn connecting the obtuse plane angles with the acute, as in *fig.* 47, they will be the short diagonals of the faces, and will coincide with the edges of a contained rhomboidal dodecahedron: which solid may thereby be made visible in the surface of the trapezohedron. The volume of the one will be to that of the other as 3 : 4.

If the equal and opposite angles be united together by lines, those lines will be the long diagonals, of the faces, and will coincide with the edges of a contained cuboçtahedron, and its angular extremities will coincide with the intermediary angular extremities of the trapezohedron. The volume of the one, will be to that of the other as 20 : 27.

The six acute angular extremities coincide with the angular extremities of a contained octahedron penetrating as it were through the surface of the trapezohedron. Their volumes are to each other as 1 : 2.

The eight obtuse angular extremities are the angular extremities of a cube, penetrating as it were through the surface of the trapezohedron. Their volumes are to each other as 3 : 8.

The angular extremities of a cube taken alternately, coincide with the angular extremities of a contained tetrahedron, consequently, the alternate obtuse angular extremities of a trapezohedron will coincide with the angular extremities of a contained tetrahedron; the contained tetrahedron will be to the trapezohedron as 1 : 8.

The angular extremities of the two contained tetrahedrons intersecting each other in the contained cube, (which is the re-octahedron, (*fig.* 30) terminate in the obtuse angular extremities of the trapezohedron. The octahedron is to the trapezohedron as 3 : 16.

The two contained tetrahedrons, by their mutual intersection, form a small or interior octahedron in the middle. This interior octahedron is to the trapezohedron as 1 : 16.

The re-octahedron may be considered as the interior octahedron, with a small tetrahedron on each of its faces, (see *fig.* 6.). One of these small tetrahedrons is to the trapezohedron as 1 : 64.

The small tetrahedron, likewise, measures all the contained simple solids having equal faces. It was just said that the trapezohedron was equal to 64 of them; the rhomboidal dodecahedron will be found equal to 48; the octahedron to 32; the cube to 24; and the tetrahedron to 8 : as 8 measures all the foregoing numbers; therefore, the large tetrahedron measures all the before mentioned solids; likewise the cube is equal to half the rhomboidal dodecahedron; and the octahedron is equal to half the trapezohedron.

The trapezohedron and its contained simple solids, viz.; the rhomboidal dodecahedron, the octahedron, the cube, and the tetrahedron, are as the number 8. 6. 4. 3. and 1. The cuboctahedron, in this case, would be equal to $5\frac{25}{27}$.

There is a simplicity of relation between the contained solids and the trapezohedron that deserves to be noticed. The rhomboidal dodecahedron and cuboctahedron have their edges visible in its surface; the rhomboidal dodecahedron has its faces at right angles to the middle axis; the cuboctahedron has one of its triangular faces, at right angles to the shortest axis, and the square faces at right angles to the longest axis. The octahedron has three faces of the trapezohedron, at right angles to each of its faces. The cube has six faces of the trapezohedron parallel to each of its axis. The tetrahedron has six faces of the trapezohedron parallel to each of its faces.

The rhomboidal dodecahedron, and cuboctahedron mutually intersect each other, as in *fig.* 70. The angular extremities of the one, protruding through the faces of the other; the angular extremities of the cuboctahedron forming rectangular pyramids, one on each face of the rhomboidal dodecahedron. The angular extremities of the rhomboidal dodecahedron, forming square and triangular pyramids on the faces of the cuboctahedron, a square pyramid on each square face, and a triangular pyramid on each triangular face: the extremities of the bases of each of these last mentioned pyramids bisect the sides of the faces on which they stand; and the extremities of the bases of the rectangular pyramids, trisect in reality the sides of the faces on which they stand, because they cut off one third: consequently, the edges of the rhomboidal dodecahedron bisects the edges of the cuboctahedron; and the edges of the cuboctahedron trisect the edges of the rhomboidal dodecahedron. When they are both contained in the same trapezohedron, as above described, they are to each other as 80 : 81, or $5 \times 2^4 : 3^4$.

The cuboctahedron has this remarkable circumstance attending it, that one of its terms expressing its relation with any one of the natural solids, in regard to their volumes, has constantly a five for one of its factors; and the other term has no greater factor than three. This will be the case, whether it be compared with any one of the natural solids contained in the same trapezohedron with itself, or whether it be compared with any one of them, supposed to be inscribed in it, or with any one of them supposed to invelope it. In each of these three cases, the greatest factor in one of the terms will be a five, and the other term will not have a greater factor than three, as will appear from the following statement of the three cases.

FIRST.—*Of the relation of the Cuboctahedron with the Natural Solids when they are contained in the same Trapezohedron with itself.*

Its relation to the rhomboidal dodecahedron contained in the same trapezohedron with itself, is as 80 : 81 or	$5 \times 2^4 : 3^4$
Its relation to the octahedron contained in the same trapezohedron is as 40 : 27 or	$5 \times 2^3 : 3^3$
Its relation to the cube contained in the same trapezohedron, is as 160 : 81 or	$5 \times 2^5 : 3^4$
Its relation to the tetrahedron contained in the same trapezohedron is as 160 : 27 or . .	$5 \times 2^5 : 3^3$

SECONDLY.—*Of its relation with the Natural Solids when they are inscribed in it.*

The cuboctahedron is to its inscribed trapezohedron as . . :	5 : 2

The cuboctahedron is to its inscribed rhomboidal dodecahedron as 10 : 3 or . . . . . . $2 \times 5 : 3$

The cuboctahedron is to its inscribed octahedron as . . . . $5 : 1$

The cuboctahedron is to its inscribed cube as 45 : 16 or . . . . $3^2\times5 : 2^4$

The cuboctahedron is to its inscribed tetrahedron as 135 : 16 or . . . . $3^3\times5 : 2^4$

Lastly.—*Of the relation of the Cuboctahedron to the Natural Solids when they envelope it.*

Its volume is to its enveloping trapezohedron as 20 : 27 or . . . . $2^2\times5 : 3^3$

It is to its circumscribing rhomboidal dodedron as 5 : 12 or . . . . $5 : 3\times2^2$

It is to its enveloping octahedron as 5 : 8 or . $5 : 2^3$

It is to its enveloping cube as 5 : 6 or . . . $5 : 2\times3$

It is to its enveloping tetrahedron as 5 : 16 or . $5 : 2^4$

In all these examples, it will be seen that one of the terms has a five for its factor, and the other term has no greater factor than three; but when the cuboctahedron is compared to another cuboctahedron, the fives disappear; as for instance, if a cuboctahedron be contained in a tetrahedron, and another cuboctahedron be circumscribed about the same tetrahedron; the two cuboctahedrons will be to each other as 1 : 27 or $1 : 3^3$; where the fives disappear, and the greatest factor is three: a similar effect takes place, if any one of the natural solids be substituted for the tetrahedron, the fives will disappear, and the greatest factor will be three.

The relation subsisting between the simple natural solids, in respect to their volumes, is equally remarkable for its simplicity; for in every instance the numbers expressing these relations are rational, and their highest factors are never greater than three.

This simplicity accompanies the relation between the remarkable lines in and about the natural solids; and though a number of those relations are expressed by surds, those surds are simple, and except in one instance, among those given, they have not a greater factor under the vinculum than three.

The acute angular extremities of a trapezohedron coincide with the angular extremities of the contained octahedron: the obtuse angular extremities coincide with the angular extremities of the contained cube and the intermediary angular extremities, with the angular extremities of a solid, that is intermidiary between the octahedron and the cube.

If a plane pass through one of the long diagonals of the trapezohedron, cutting the solid, at right angles to its face, it will divide it into two equal parts, and the face of the section will be a regular hexagon. This section will divide the contained rhomboidal dodecahedron, the cuboctahedron, the octahedron, the cube, and the re-octahedron, into two equal parts, and the faces of all their sections will be hexagonal; the four first will be regular hexagons, and the last a star with six points; this section will be at right angles to the shortest axis and will bisect that axis, it will likewise cut six of the small tetrahedrons in an equal and similar manner; and the interior octahedron in the middle will be divided into two equal parts, the face of whose section will be the small hexagon, in the middle of *fig.* 73, and the faces of the sections of the small tetrahedrons, will be the equilateral triangles surrounding that small hexagon. The contained cube and octahedron in a trapezohedron, intersect each other in a similar manner as in the rhomboidal dodecahedron, forming salient and re-entering angles; this section passing through the bottom of the re-entering angles, cuts none of the protruded parts of either of the solids; the hexagon surrounding the small triangles, belongs equally to the cube and octahedron. The hexagon circumscribing this, is the section of the contained rhomboidal dodecahedron, as the outermost hexagon is the section of the contained cuboctahedron. In this section of the trapezohedron, no part of it is cut that is not common to some one of the contained solids, notwithstanding there is no part of its surface, with the exception of the diagonals, common to any of them.

If a plane pass through the longest edges of a trapezohedron, at right angles to its longest axis, it will cut the solid into two equal parts, and likewise all the contained solids; making the faces of the section of the contained solids, squares in every instance; which section is represented, *fig.* 74, where the section of the interior octahedron, and the ro-octahedron coincide and are represented by the small square in the middle; in this section none of the small tetrahedrons are cut: the section of the cube is the square circumscribing the small one, and the section of the octahedron circumscribes that; and because the section passes through the edges of the octahedron, and the edges of the octahedron coincide with the long diagonals of the rhomboidal dodecahedron, therefore this section cuts no part of the last mentioned solid that is not common to the octahedron; the outer square is the section of the cub-octahedron, the four corners of the square are the sections of four rectangular pyramids, protruding through the surface of the rhomboidal dodecahedron, *fig.* 71: the extremities of the large square are the sections of the square pyramids: this section passes through the perpendiculars drawn on the faces of the pyramids, which lines coincide with the edges of the contained octahedron. The octagon circumscribing these squares, is the section of the trapezohedron; the triangular spaces are the sections of those parts of the trapezohedron, not common to any of the contained solids.

A plane passing through the obtuse angular extremities, *c d e*, of a trapezohedron, *fig*. 29, coincides with the face of a contained tetrahedron, and bisects six of the longest edges of the trapezohedron as *f b* in *a*.

A plane passing through the obtuse angular extremities *a b c d*, (*fig*. 35) coincides with the faces of a contained cube; this plane cuts off one fourth from four of the longest edges of the trapezohedron, as *e r*, which is one fourth, *r s*.

A plane passing through the three acute angular extremities, *a b e*, (*fig*. 41), coincides with the face of the contained octahedron, and cuts off two fifths from three of the shortest edges of the trapezohedron, as *r n* which is two fifths *r s*.

To make a model to represent *fig*. 71, take a rhomboidal dodecahedron, whose edges are equal to the short diagonals of a trapezohedron; and a cuboctahedron, whose edges are equal to the long diagonals; mark off one third from each edge of the rhomboidal dodecahedron, measuring from the obtuse angular extremities, and draw lines uniting the adjacent points of bisection; these lines will be parallel to the diagonals of the faces: the obtuse angular extremities will become triangular pyramids, the acute angular extremities will become square pyramids, and each face will have a rectangle described on it by these lines, each of which rectangles will be equal to the base of a rectangular pyramid; their sides are to each other as 1 : $\sqrt{2}$, the same as the two diagonals of the face on which they are drawn. The edges of the cuboctahedron must be all bisected, and lines must be drawn connecting the adjacent points of bisection; these lines will form a triangle on the middle of each triangular face, and a square in the middle of each square face: the angular extremities marked off by them will become rectangular pyramids, which pyramids, if they were removed, and placed on the rectangular spaces marked off on the faces of the rhomboidal dodecahedron, would convert it into an exact representation of *fig.* 71, or if the triangular and square pyramids were removed, from the rhomboidal dodecahedron and placed on the triangular and square spaces, marked off on the faces of the cuboctahedron, they would produce exactly the same effect.

OF THE LONGEST AXIS.

A plane passing through the acute solid angle at right angles to the longest axis, bisects that axis, if it pass through an obtuse solid angle at right angles to the same axis, it cuts off one fourth: if it pass through an intermediary solid angle at right angles to the same axis, it cuts off one sixth.

OF THE SHORTEST AXIS.

If a plane pass through one of the long diagonals of the face of a trapezohedron at right angles to the shortest axis farthest distant, it will bisect that axis; if it pass through an obtuse angular extremity at right angles, to the same axis, it will cut off one third; if it pass through an acute angular extremity at right angles, to the same axis, it will cut off one sixth: and through an intermediary angular extremity next above, at right angles to the same axis, it will cut off one eighteenth.

OF THE MIDDLE AXIS.

A plane passing through the short diagonal, at right angles to the middle axis farthest distant, bisects that axis; if it pass through an intermediary solid angle, next above at right angles, to the same axis, it cuts off one fourth: if it pass through an acute or an obtuse angular extremity, next above, at right angles to the same axis, it cuts off one eighth.

The faces of the section made by a plane, bisecting the middle axis of a trapezohedron at right angles, are bounded by some remarkable lines in or about the face of the solid to which the section belongs; the face of the section of the trapezohedron is bounded by four of the short sides and four of the short diagonals of its faces. The face of the section of the contained rhomboidal dodecahedron is bounded by four sides and two short diagonals of its faces. The face of the section of the contained octahedron is bounded by four perpendiculars of its faces. The face of the section of the contained cube is bounded by two edges and two diagonals of its faces: and the face of the section of one of the contained tetrahedrons is bounded by one edge and two perpendiculars of its faces.

If all the short diagonals of the faces of a trapezohedron be drawn, they will divide the surface of the solid into twelve quadrangular pyramids; one on each face of the contained rhomboidal dodecahedron. If lines be drawn from the angular extremities of the contained rhomboidal dodecahedron to its centre, they will divide it into twelve quadrangular pyramids, which will have an inverted position; each of these inverted pypyramids will be a twelfth part of the contained rhomboidal dodecahedron, and each of the first mentioned pyramids will be a twelfth part of the excess of the trapezohedron above the contained rhomboidal dodecahedron; consequently, the relation between these two solids may be determined by the relation between these two pyramids: but the base of these pyramids being common to both, they will be to each other as their altitudes; therefore, the relation between the volume of the trapezohedron, and its contained rhomboidal dodecahedron may be determined by the relation between the altitude of these two pyramids.—*See demonstrations.*

The altitude of the contained tetrahedron is to the shortest axis of the trapezohedron as . . . . . 2 : 3

The altitude of the contained cube is to the longest axis of the trapezohedron as . . . . 1 : 2

The altitude of the contained octahedron, is to the shortest axis of the trapezohedron as . . . . 2 : 3

The altitude of the contained rhomboidal dodecahedron is to the middle axis of the trapezohedron as 3 : 4 or. . . . . $3 : 2^2$

The contained tetrahedron is to the trapezohedron as 1 : 8 or . . . . $1 : 2^3$

The contained cube is to the trapezohedron as 3 : 8 or . . . . $3 : 2^3$

The contained octahedron is to the trapezohedron as . . . . 1 : 2

The contained rhomboidal dodecahedron is to the trapezohedron as 3 : 4 or . . . . $3 : 2^2$

If a trapezohedron be contained in a tetrahedron, a cube, an octahedron, or a rhomboidal dodecahedron, and another trapezohedron envelope each of these solids; the contained trapezohedron will be to the enveloping trapezohedron in the case of the intervening

Tetrahedron as 1 : 27 or . . . . $1 : 3^3$
Cube as 1 : 8 or . . . . $1 : 2^3$
Octahedron as 8 : 27 or . . . . $2^3 : 3^3$
Rhomboidal Dodecahedron as 27 : 64 or : . . $3^3 : 2^6$

If the edges of the trapezohedron be removed in a direct manner, it will be converted into a solid, having twenty four rhombic faces grouped together so as to form eight very obtuse rhombohedrons in the surface of the new solid, and twenty four trapezoidal faces grouped together in such a manner so as to form stars with four points between the rhombohedrons. Hence the trapesohedron is the last simple solid in this series, or the last solid whose faces are all equal and similar.

The farther consideration of the relations of the dodecahedron and the icosahedron, and the solids to be derived from them, will be resumed from page 29.

THE TRIACONTAHEDRON.

This solid may be formed from a dodecahedron or an icosahedron; in a similar manner, as the rhomboidal dodecahedron was formed from the cube and octahedron. It has

Thirty faces,
Sixty edges,
Thirty-two angular extremities,
One hundred and twenty plane angles, and
Sixteen axes.

Its faces are all equal and similar rhombs. Its edges are all equal. The solid angles are of two varieties; twelve acute, each of which is formed by the meeting of five acute plain angles; and twenty obtuse, each of which is formed by the meeting of three obtuse plane angles. The plane angles are of two varieties, sixty obtuse and sixty acute. The axes are of two varieties, six long and ten short; the extremities of each of the long axes terminate, in the two opposite acute angular extremities of the solid, and the extremities of the short axes terminate in its opposite obtuse angular extremities.

The inclination of the face united by its edges is 144°. The inclination of the faces with the edge crossing the obtuse solid angles is 148° 16′ 57″. The inclination of the face with the opposite edge crossing the acute solid angle is 121° 43′ 3″. The inclination of the edge with the longest axis 63° 26′ 6″. The inclination of the edge with the shortest axis is 79° 11′ 15″½. The inclination of the face with the shortest axis is 69° 5′ 41″½. The inclination of the face with the longest axis is 58° 16′ 57″.

The long diagonal is to the short as $\sqrt{5}+1 : 2$ or as $2 : \sqrt{5}-1$.

This solid, as was before observed, may be generated from a dodecahedron or an icosahedron, if the edges of either of the solids were removed by planes, passing through the centre of the faces adjoining to the edge to be removed in a direction parallel to the edge.

There is but one variety that can be made, which is bounded by thirty equal and similar rhombs; which admits of no variations like the trapezohedron.

There is no solid that can be formed, of a greater number of equal and similar rhombs united together, so that the obtuse plane angles should meet to form one set of angular extremities, and the acute plane angles should meet to form another set of angular extremities.—*See demonstrations.*

If all the long diagonals of the triacontahedron be drawn, they will coincide with the edges of the contained icosahedron; and the angular extremities of the contained icosahedron will terminate in its acute angular extremities. If all the short diagonals be drawn, they will coincide with the edges of the contained dodecahedron; and the angular extremities of the contained dodecahedron will terminate in its obtuse angular extremities.

The icosahedron and dodecahedron mutually intersect each other in the triacontahedron; in a similar manner, as the cube and octahedron intersect each other in the rhomboidal dodecahedron; their edges cross each other at right angles, and they are in the same plane. If an icosahedron be taken, whose edges are equal to the long diagonals of a triacontahedron; and a dodecahedron be taken, whose edges are equal to the short diagonals of the same triacontahedron; and the edges of both

the icosahedron and dodecahedron be bisected, and lines be drawn on the faces of both solids, uniting the adjacent points of bisection; the surface of the dodecahedron will be divided into twenty triangular pyramids, and twelve pentagonal spaces; and the surface of the icosahedron will be divided into twelve pentagonal pyramids and twenty triangular spaces: If the pyramids belonging to one of the solids be removed and placed on the spaces marked off on the faces of the other solid; it will be a perfect representation of the icosahedron and decahedron intersecting each other in the triacontahedron; gular extremities of one protruding through the faces of her, similar to the cube and octahedron in a rhomboidal cahedron.

a line be drawn through the centre of the face of a ntahedron, at right angles to one of its edges; and a plane hrough this line and cut the solid at right angles to the face ich the line is drawn, it will bisect the longest axis, and ce of the section will be a regular decagon; the faces of the ns of all the contained solids will be also regular decagons. section divides the contained icosahedron and dodecahe- into two equal parts without cutting any of their protruded nities represented above, on account of its passing along des of the bases of the pyramids.

plane pass through the centre of a face, at right angles shortest axis, the face of the section will have twelve ; consequently, the number of sides of this section are ble by six, which is the case with the faces of the sections l the contained solids made with this section; the number sides of each section is divisible by six.

If all the long diagonals be drawn on the face of a triaconta-

The long diagonal is to the short as $\sqrt{5}+1 : 2$ or as $2 : \sqrt{5}-1$.

This solid, as was before observed, may be generated from a dodecahedron or an icosahedron, if the edges of either of the solids were removed by planes, passing through the centre of the faces adjoining to the edge to be removed in a direction parallel to the edge.

There is but one variety that can be made, which is bounded by thirty equal and similar rhombs; which admits of no variations like the trapezohedron.

There is no solid that can be formed, of a greater number of equal and similar rhombs united together, so that the obtuse plane angles should meet to form one set of angular extremities, and the acute plane angles should meet to form another set of angular extremities.—*See demonstrations.*

If all the long diagonals of the triacontahedron be drawn, they will coincide with the edges of the contained icosahedron; and the angular extremities of the contained icosahedron will terminate in its acute angular extremities. If all the short diagonals be drawn, they will coincide with the edges of the contained dodecahedron; and the angular extremities of the contained dodecahedron will terminate in its obtuse angular extremities.

The icosahedron and dodecahedron mutually intersect each other in the triacontahedron; in a similar manner, as the cube and octahedron intersect each other in the rhomboidal dodecahedron; their edges cross each other at right angles, and they are in the same plane. If an icosahedron be taken, whose edges are equal to the long diagonals of a triacontahedron; and a dodecahedron be taken, whose edges are equal to the short diagonals of the same triacontahedron; and the edges of both

the icosahedron and dodecahedron be bisected, and lines be drawn on the faces of both solids, uniting the adjacent points of bisection; the surface of the dodecahedron will be divided into twenty triangular pyramids, and twelve pentagonal spaces; and the surface of the icosahedron will be divided into twelve pentagonal pyramids and twenty triangular spaces: If the pyramids belonging to one of the solids be removed and placed on the spaces marked off on the faces of the other olid; it will be a perfect representation of the icosahedron and hedron intersecting each other in the triacontahedron; ar extremities of one protruding through the faces of similar to the cube and octahedron in a rhomboidal dron.

ne be drawn through the centre of the face of a edron, at right angles to one of its edges; and a plane gh this line and cut the solid at right angles to the face the line is drawn, it will bisect the longest axis, and the section will be a regular decagon; the faces of the all the contained solids will be also regular decagons. on divides the contained icosahedron and dodecahe- wo equal parts without cutting any of their protruded represented above, on account of its passing along f the bases of the pyramids.

e pass through the centre of a face, at right angles test axis, the face of the section will have twelve sequently, the number of sides of this section are six, which is the case with the faces of the sections ined solids made with this section; the number section is divisible by six.

gonals be drawn on the face of a triaconta-

The long diagonal is to the short as $\sqrt{5}+1 : 2$ or as $2 : \sqrt{5}-1$.

This solid, as was before observed, may be generated from a dodecahedron or an icosahedron, if the edges of either of the solids were removed by planes, passing through the centre of the faces adjoining to the edge to be removed in a direction parallel to the edge.

There is but one variety that can be made, which is bounded by thirty equal and similar rhombs; which admits of no variations like the trapezohedron.

There is no solid that can be formed, of a greater number of equal and similar rhombs united together, so that the obtuse plane angles should meet to form one set of angular extremities, and the acute plane angles should meet to form another set of angular extremities.—*See demonstrations.*

If all the long diagonals of the triacontahedron be drawn, they will coincide with the edges of the contained icosahedron; and the angular extremities of the contained icosahedron will terminate in its acute angular extremities. If all the short diagonals be drawn, they will coincide with the edges of the contained dodecahedron; and the angular extremities of the contained dodecahedron will terminate in its obtuse angular extremities.

The icosahedron and dodecahedron mutually intersect each other in the triacontahedron; in a similar manner, as the cube and octahedron intersect each other in the rhomboidal dodecahedron; their edges cross each other at right angles, and they are in the same plane. If an icosahedron be taken, whose edges are equal to the long diagonals of a triacontahedron; and a dodecahedron be taken, whose edges are equal to the short diagonals of the same triacontahedron; and the edges of both

the icosahedron and dodecahedron be bisected, and lines be drawn on the faces of both solids, uniting the adjacent points of bisection; the surface of the dodecahedron will be divided into twenty triangular pyramids, and twelve pentagonal spaces; and the surface of the icosahedron will be divided into twelve pentagonal pyramids and twenty triangular spaces: If the pyramids belonging to one of the solids be removed and placed on the spaces marked off on the faces of the other solid; it will be a perfect representation of the icosahedron and dodecahedron intersecting each other in the triacontahedron; angular extremities of one protruding through the faces of other, similar to the cube and octahedron in a rhomboidal decahedron.

If a line be drawn through the centre of the face of a acontahedron, at right angles to one of its edges; and a plane s through this line and cut the solid at right angles to the face which the line is drawn, it will bisect the longest axis, and face of the section will be a regular decagon; the faces of the tions of all the contained solids will be also regular decagons. is section divides the contained icosahedron and dodecahe- on into two equal parts without cutting any of their protruded remities represented above, on account of its passing along sides of the bases of the pyramids.

If a plane pass through the centre of a face, at right angles the shortest axis, the face of the section will have twelve es; consequently, the number of sides of this section are visible by six, which is the case with the faces of the sections all the contained solids made with this section; the number the sides of each section is divisible by six.

If all the long diagonals be drawn on the face of a triaconta-

hedron, they will not only coincide with the edges of the contained icosahedron, but will also divide the surface into twenty triangular pyramids: if at the same time, lines be drawn from the angular extremities of the contained icosahedron to its centre; it will be divided into twenty triangular pyramids, which will be in an inverted position to the first mentioned pyramids: the two pyramids taken together will be a twentieth part of the triacontahedron. The first mentioned pyramids will be a twentieth part of the excess of the triacontahedron, above the contained icosahedron, and the last mentioned pyramids will be a twentieth part of the contained icosahedron; consequently, the volume of the contained icosahedron will be to the triacontahedron, as the longest pyramid is to the two pyramids taken together; but the two pyramids have a common base, therefore they are to each other as their altitudes. Hence whatever ratio the two altitudes taken together as one line and the longest altitude have to each other, that will be the ratio between the two solids.

If all the short diagonals of the triacontahedron be drawn they will not only coincide with the edges of the contained dodecahedron, but will also divide the surface of the triacontahedron into twelve pentagonal pyramids, one on each face of the contained dodecahedron. If the contained dodecahedron have lines drawn from each of its angular extremities to its centre; it will be divided by these lines into twelve pentagonal pyramids, which will have an inverted position to the former pyramids, and their bases will be common to both: therefore the volumes of the two pyramids will be to each other as their altitudes. The two pyramids taken together will be a twelfth part of the triacontahedron, and the inverted pyramid will be a twelfth part of the contained dodecahedron; consequently these two solids will be to each other, as the altitudes of the two pyramids taken together are to the altitude of the inverted pyramid.—*See the Appendix.*

THE HEXACONTAHEDRON.

This solid may be formed from the triacontahedron in a similar manner, as the trapezohedron is formed from the rhomboidal dodecahedron. It has

Sixty faces,
One hundred and twenty edges,
Two hundred and forty plane angles,
Sixty-two angular extremities, and
Thirty-one axes.

Its faces are all equal and similar. It has two varieties of edges, sixty long and sixty short. It has three varieties of plane angles, sixty obtuse, sixty acute, and one hundred and twenty acute nearly approaching a right angle. It has three varieties of axes, six long, fifteen middle, and ten short.

Its obtuse angles are 119° 6′ 26″.

Its acute plane angles are 66° 2′ 1″¼.

Its less acute angles are 87° 25′ 46″.

The inclination of the faces united by the least acute plane angles are 144°. The inclination of the faces with the short-edges crossing the obtuse angular extremities is 163° 48′ 58″½. The inclination of the faces with the opposite longest edges crossing the acute angular extremities is 131° 24′ 40″½. The inclication of the faces with the longest axis is 63° 26′ 6″. The inclination of the faces with the middle axis is 72°. The inclination of the faces with the shortest axisis 70° 11′ 15″½ The inclination of the longest edges crossing the intermediary angular extremities is $160_{\circ}$ 36′ 45″. The mutual inclination of the shortest edges crossing the intermediary angular extremities is 149° 5′ 57″

The long diagonal is to the short as $\sqrt{25+5\sqrt{5}} : 32$. The longest edge is to the shortest edge as $\sqrt{37+15\sqrt{5}} : \sqrt{17+5\sqrt{5}}$. The longest diagonal bisects the shortest, and the short diagonal divides the long into two parts, which parts are to each other as $2 : 3+\sqrt{5}$ or as $3-\sqrt{5} : 2$. The longest axis is to the shortest as $\sqrt{5+\sqrt{5}} : \sqrt{6}$. The middle axis is to the shortest axis as $4 : \sqrt{15} :: \sqrt{16} : \sqrt{15}$.

The middle axis of the hexacontahedron is to the altitude of the triacontahedron $2\sqrt{5}+6 : 2\sqrt{5}+5$.

The hexacontahedron may be formed from a triacontahedron, by removing its edges by planes passing through the centres of the faces adjoining to the edge to be removed, in a direction parallel to that edge.

The faces of this solid are very similar to the faces of the trapezohedron and are united together in a similar manner. Like the trapezohedron, its angular extremities are formed by or point out, the angular extremities of the contained solids.

If the long diagonals of a hexacontahedron be drawn, they will coincide with the edges of a contained triacontahedron, whose angular extremities will terminate in the acute angular extremities of the hexacontahedron, and its obtuse angular extremities in the obtuse angular extremities of the hexacontahedron.

The acute angular extremities of the hexacontahedron coincide, with the angular extremities of the contained icosahedron; and the obtuse angular extremities coincide with the angular extremities of the contained dodecahedron. The icosahedron and dodecahedron mutually intersect each other in a similar manner, as they do in the triacontahedron, being themselves contained in the contained triacontahedron.

If all the short diagonals be drawn on the face of a hexacontahedron they will coincide with the edges of a contained Icosadodecahedron whose angular extremities terminate in the intermediary angular extremities of the hexacontahedron. Hence the acute angular extremities coincide with the angular extremities of the contained icosahedron, and the obtuse angular extremities with the angular extremities of a contained dodecahedron, and the intermediary angular extremities, with the angular extremities of a contained intermediary solid, viz. the Icosadodecahedron.

The triacontahedron and icosadodecahedron, intersect each other in a similar manner as the rhomboidal dodecahedron and cuboctahedron intersect each other in the trapezohedron, the angular extremities of one protruding through the faces of the other, and as the edges of the rhomboidal dodecahedron, cross each other at right angles in the surface of the trapezohedron, so the edges of the contained triacontahedron and cuboctahedron cross each other in the surface of the hexacontahedron.

If through one of the diagonals, that unites the equal plane angles, a plane pass at right angles to the faces on which the diagonal is drawn; it will bisect one of the longest axis at right angles, and divide the solid into two equal parts, making the face of the section a regular decagon, represented in *fig.* 53. The outer decagon is in reality, a section of the contained icosadodecahedron, because this section cuts no part of the hexacontahedron that is not common to the contained solids, but passes through ten of the diagonals of its faces, which diagonals coincide with the edges of the contained icosadodecahedron. The second decagon is the section of the contained triacontrahedron, which section cuts its faces at right angles to its edges. The third decagon is the section of the contained icosahedron and dodecahedron represented above, as intersecting each other; this section cuts none of their protruded parts, but passes through the line of junction between the two kinds of pyramids, formed by the angular extremities of the two solids protruding through each other's faces, consequently, the face of this section is common to both. The two pentagons, intersecting each other in the middle, will be explained when the pentahedron is described.

If a plane cut the solid into two equal parts, at right angles to the shortest axis; the number of sides by which the section of the hexacontahedron is bounded, and likewise the sections of all the contained solids, will be divisible by six, whereas, in the former section they are divisible by five.

If a plane cut the hexacontahedron into two equal parts at right angles to the middle axis, the face of the section, and likewise the faces of the sections of all the contained solids will be bounded by some of the most remarkable lines, of the faces of the different solids of which they are the sections. The face of the section of the hexacontahedron will be bounded by twelve sides, four of which will be the longest sides of the face, four will be the shortest sides, and the remaining four will be the longest diagonals of the face. The section of the contained triacontahedron will be bounded by eight sides, two of which will be the longest diagonals of its faces, two of the shortest diagonals and the remaining four will be the edges of its face. The face of the sections of the contained icosahedron will be bounded by six sides, four of which will be the perpendiculars of the faces, and the remaining two will be the sides of the faces. The section of the contained dodecahedron will be bounded by four perpendiculars of the faces and two edges.

Two solids have been described, . the trapezohedron and the hexacontahedron, bounded by equal and similar trapezoidal faces united together in a similar manner. Each of these admit of an indefinite variety; the plane angles of the faces being capable of great variations, without altering their general form, by elongating or shortening the longest or shortest axes; and supposing each face to move in an equal manner, on that diagonal which unites the equal plane angles, as on an axis. There is another solid capable of being formed with twelve trapezoidal faces united together in a similar manner, named the trapezoidal dodecahedron, which has a tetrahedron for its base, whereas the trapezohedron has an octahedron, and the hexacontahedron has an icosahedron for their bases, like the two last mentioned solids, it admits of indefinite variations. Besides these three, there is no other solid that can be formed of similar trapezoidal faces, united together in a similar manner.

If all the long diagonals be drawn on the faces of a hexacontrahedron, they will not only coincide with the edges of

The long diagonal is to the short as $\sqrt{5}+1:2$ or as $2:\sqrt{5}-1$.

This solid, as was before observed, may be generated from a dodecahedron or an icosahedron, if the edges of either of the solids were removed by planes, passing through the centre of the faces adjoining to the edge to be removed in a direction parallel to the edge.

There is but one variety that can be made, which is bounded by thirty equal and similar rhombs; which admits of no variations like the trapezohedron.

There is no solid that can be formed, of a greater number of equal and similar rhombs united together, so that the obtuse plane angles should meet to form one set of angular extremities, and the acute plane angles should meet to form another set of angular extremities.—*See demonstrations.*

If all the long diagonals of the triacontahedron be drawn, they will coincide with the edges of the contained icosahedron; and the angular extremities of the contained icosahedron will terminate in its acute angular extremities. If all the short diagonals be drawn, they will coincide with the edges of the contained dodecahedron; and the angular extremities of the contained dodecahedron will terminate in its obtuse angular extremities.

The icosahedron and dodecahedron mutually intersect each other in the triacontahedron; in a similar manner, as the cube and octahedron intersect each other in the rhomboidal dodecahedron; their edges cross each other at right angles, and they are in the same plane. If an icosahedron be taken, whose edges are equal to the long diagonals of a triacontahedron; and a dodecahedron be taken, whose edges are equal to the short diagonals of the same triacontahedron; and the edges of both

the icosahedron and dodecahedron be bisected, and lines be drawn on the faces of both solids, uniting the adjacent points of bisection; the surface of the dodecahedron will be divided into twenty triangular pyramids, and twelve pentagonal spaces; and the surface of the icosahedron will be divided into twelve pentagonal pyramids and twenty triangular spaces: If the pyramids belonging to one of the solids be removed and placed on the spaces marked off on the faces of the other solid; it will be a perfect representation of the icosahedron and dodecahedron intersecting each other in the triacontahedron; e angular extremities of one protruding through the faces of e other, similar to the cube and octahedron in a rhomboidal odecahedron.

If a line be drawn through the centre of the face of a iacontahedron, at right angles to one of its edges; and a plane ass through this line and cut the solid at right angles to the face which the line is drawn, it will bisect the longest axis, and e face of the section will be a regular decagon; the faces of the ections of all the contained solids will be also regular decagons. his section divides the contained icosahedron and dodecahe-ron into two equal parts without cutting any of their protruded xtremities represented above, on account of its passing along he sides of the bases of the pyramids.

If a plane pass through the centre of a face, at right angles the shortest axis, the face of the section will have twelve ides; consequently, the number of sides of this section are ivisible by six, which is the case with the faces of the sections f all the contained solids made with this section; the number f the sides of each section is divisible by six.

If all the long diagonals be drawn on the face of a triaconta-

A variety of curious relations subsisting between the four solids first mentioned, have been discovered by Dr. Roget, and are given in the Appendix.

The dodecahedron, icosahedron, triacontahedron, and hexacontahedron, have been shown to possess similar properties between themselves and to have a similar relation to each other, as the cube, octahedron, rhomboidal dodecahedron and trapezohedron have, forming two distinct series; the first series, however, is deficient for want of a solid that has the same relation to it that the tetrahedron has to the last. To ascertain the form of this solid, the re-octahedron and the re-dodecahedron will be found of essential service; the pentagonal pyramids of the re-dodecahedron having the same relation to the dodecahedron, as the tetrahedron of the re-octahedron has to its octahedron. This pentagonal pyramid may be called a pentahedron to distinguish it from all other pentagonal pyramids.

THE PENTAHEDRON.

This solid is a pentagonal pyramid, whose base is a regular pentagon, and whose sides are equal and similar isosceles triangles; having the angles at the base, each double the third angle.

The pentahedron will be found, in some respects, more curious and interesting than the tetrahedron, it was before noticed that the icosahedron was in some of its properties analogous to the cube and the dodecahedron to the octahedron, which circumtsance will be confirmed by the comparison of the pentahedron and tetrahedron.

It may be seen, (*fig* 6.) that two tetrahedrons, mutually intersecting each other, form an octahedron in the middle, which is that part of each tetrahedron which is common to both, a similar effect takes place, when two pentahedrons mutually intersect each other, as in *fig*. 12, where, by their mutual intersection, they form a dodecahedron, which is that part of each pentahedron which is common to both.

If an octahedron be placed on one of its sides, while it is in that position, it may be said to have an upper and lower base and six lateral faces; but these lateral faces are of two kinds, three inferior and three superior; if the three superior lateral faces be extended in every direction, and at the same time, the lower base of the octahedron; these four planes will intersect each other, forming a tetrahedron like *fig*. 25.

In like manner, a dodecahedron being placed on one of its sides, that side will then be the lower base, and the face opposite will be the upper base; the remaining ten faces will be lateral faces, five of which will be inferior, aud five superior; if the superior lateral faces be extended above and below, and, at the same time, the lower base, as was done with the octahedron, till they intersect each other; these six planes, by their mutual intersection, will form a pentahedron.

The inclination of the superior edge of a tetrahedron on its face, in a direction crossing its summit, is equal to the inclination of the same edge on its base; this is the case with the pentahedron.

It may be seen (*fig.* 25.) that one half of the faces of a contained octahedron may be marked out on the faces of a tetrahedron, in almost as simple, and much more curious manner, one half of the faces of a contained dodecahedron, may be marked out on the faces of a pentahedron.

If the edge *A B* of a pentahedron (*fig* 54,) be bisected in *a*, and from the point of bisection, the perpendicular *a b* be drawn on the face *A B C*, and *b B* be joined, and the same be done with the edge *A C*, and *d C* be joined, and *a e* be taken equal to *a d*, and through *e*, *e n* be drawn parallel to *C B*, the pentagon *e n b r d* will be a regular pentagon, and will be one of the faces of the contained dodecahedron. In a similar manner, four other faces of the contained dodecahedron may be traced out on the remaining sides of the pentahedron. The sixth must be traced out on its base, which is done by drawing lines connecting the alternate extremities of the base, making a star similar to *fig*. 60. The small pentagon in the middle, is a face of the contained dodecehedron, making altogether six faces, traced out on the surface of the pentahedron, none of which are opposite or parallel to each other; and because one half of the faces of a dodecahedron is opposite and parallel to the other half, therefore, the remaining half of the faces of the contained dodecahedron, will be oppesite and parallel to the faces traced out on the surface of the pentahedron; and consequently, they will lay concealed, one under each of its angular extremities; if these angular extremities, which are the five extremities of the base, together with the apex, be removed by planes parallel to the opposite faces, at a distance equal to the distance of the section *C d E*, *fig*. 54, from the face opposite, which section cuts off the angular extremity *B*, by passing through the adjoining extremities, *C* and *E*, parallel to the face opposite to *B*. These six sections will convert the pentahedron into a dodecahedron. The distance between the section *C d E*, from the face opposite will be the altitude of the contained dodecahedron. This relation of the dodecahedron, is similar to that which subsists between a tetrahedron and its contained octahedron, for one half of the faces of the contained octahedron may be traced out on the faces of the tetrahedron, none of which are opposite or parallel, consequently, the remaining half of the faces of the contained octahedron, will be opposite and parallel to those traced out, and will lay concealed under the angular extremities of the tetrahedron, which extremities, if removed by planes parallel to the opposite faces, at an equal distance, which distance must be such as to cause the planes to bisect the edges of the tetrahedron. These sections will convert the tetrahedron into an octahedron.

It must be noticed that there is a part of the pentahedron which does not properly belong to it, which is the irregular tetrahedron, *B C r o fig.* 54 and *fig.* 55. If these parts of the pentahedron be removed, a pentahedron proper will be obtained represented in an inverted position by *fig.*18, which, though it appears more complicated yet its analogy to the tetrahedron will be traced with more ease, for now its solid angles are all equal, and as the tetrahedron may be considered as an octahedron with half as many tetrahedrons enveloping it, as it has faces, so in like manner the pentahedron proper may be considered as a dodecahedron, with half as many pentahedrons enveloping it as it has faces ; or, as the tetrahedron may be considered as an octahedron standing on one of its faces, having a small tetrahedron on its upper face and one on each inferior lateral face—*(see* *fig.* 25), so the pentahedron proper may be considered as a dodecahedron placed on one of its faces, with a pentahedron on its upper face, and one on each of its inferior lateral faces.

In the re-dodecahedron, *fig.* 24, there may be twelve pentahedrons proper traced out, for each solid may be considered as the summit of one, and the five adjacent angular extremities will be the angular extremities of the base of another, and the opposite angular extremity will be its summit, whereas, the five remaining or alternate angular extremities will be the extremities of the base of the one first mentioned, so that they will appear to intersect each other in pairs. A similar effect actually takes place in the re-octahedron, for as each angular extremity of a tetrahedron may be considered as the summit of a triangular pyramid, of which the opposite face is the base, therefore, each solid angle of the re-octahedron may be considered as the summit of a triangular pyramid, consequently, eight triangular pyramids may be traced out in the re-octahedron intersecting each other in pairs; but, on account of the peculiar symetry of the re-octahedron, this, as well as other effects, are obscured, owing to its having the appearance of two large tetrahedrons intersecting each other.

The pentahedron has the same relation to the icosahedron as the tetrahedrcn has to the cube, for if *A* (*fig.* 36,) be the summit of a contained tetrahedron, *G H F* the lower lateral angles, will be the extremities of its base. In like manner, of *A*, (*fig.* 57) be the summit of a contained pentahedron *C*, *D*, *E*, *F*, *d*, the lower lateral angles will be the extremities of the base of the contained pentahedron, and as each angular extremity of the cube may be considered, as the summit of a contained tetrahedron; so each angular extremity of an icosahedron may be considered, as the summit of a contained pentahedron. In like manner each acute angular extremity of a triacontahedron, or of a hexacontahedron may be considered as the summit of a contained pentahedron, from which it is easy to trace out, the lower extremities. The angular extremities of an icosahedron, and the acute angular extremities of a triacontahedron, or hexacontahedron may be considered as the extremities of two pentahedrons, mutually intersecting each other as *fig*. 12. The two pentagons in the section of the hexacontahedron, *fig*. 53, is the section of the two pentahedrons, *fig*. 12, or the section of the two pentahedrons proper, mutually intersecting each other (*fig*. 24.)

What has been said of the pentahedron is sufficient to prove, that it possesses a similar relation to the dodecahedron, the icosahedron, the triacontahedron, and the hexacontahedron; as the tetrahedron has to the cube, the octahedron, the rhomboidal dodecahedron and the trapezohedron; and from what has been said of the series commencing with the pentahedron, it must be evident that it is parallel to the natural series, in a number of its properties. As none of the solids, composing the series commencing with the pentahedron, have ever been found in nature, and as they are entirely the work of art, it may be called the artificial series, to distinguish it from the natural series.

Hence it may be concluded, that the dodecahedron and the icosahedron, not only form part of a continued series, under the appellation of the Platonic bodies, but occupy a position in a distinct series, parallel to the cube and octahedron.

The natural series will admit of indefinite variations, from the variations of the cube; which is the most simple generating solid, or the solid from which all the others may be generated, by the most simple means. This will appear evident upon inspecting *figs.* 26, 32, 38 and 44, and attending to what has been said respecting them; hence it may with propriety be called the generating solid, not only of the natural series, but of all other solids, there being no solid from which all other solids could be derived, or to which they could be referred, with similar ease.

If the cube be considered as a parallelopiped, and the different varieties of parallelopipeds be successively substituted in its place, there may be generated from each parallelopiped a different tetrahedron, octahedron, dodecahedron, and trapezohedron.

If the octahedrons, which are derived from those parallelopipeds called right and oblique quadrangular prisms, be described, some idea may be formed of the varieties that might be produced.

If a right quadrangular prism, with a square base, have all the diagonals of its faces, drawn in every direction, and each of its solid angles be removed, by planes passing through three adjacent diagonals, these planes by their mutual intersection, will form an octahedron in the middle of the prism, whose angular extremities will terminate in the centres of its faces. If the altitude of the prism be less than the side of the base, it will be a flattened or obtuse octahedron, having one of its axis shortened; if the sides be equal, it will be a cube, and the octahedron will be the regular octahedron; but if the sides of the prism be longer than the sides of the base, the octahedron will be an acute or elongated octahedron, having one of its axis elongated.

If a right quadrangular prism, with a rectangular base, have all its solid angles removed in a similar manner; the octahedron thence arising will be a rhombic octahedron, or an octahedron the bases of whose pyramids will be rhombs, its three axes will be all different, having the same proportion to each other as the edges of the prism.

If a right quadrangular prism, having a rhombic base, have all its solid angles removed in a similar manner; the octahedron thence arising will be a rectangular octahedron, or an octahedron which may be divided into two rectangular pyramids.

If a right quadrangular prism, having a rhomboidal base, have all its solid angles removed in a similar manner; the octahedron thus produced will be capable of being divided, into two quadrangular pyramids having rhomboidal bases.

For every distinct variety of parallelopiped, there will arise a distinct variety of octahedron; but it must be noticed, that every distinct variety of quadrangular prism is not a distinct variety of parallelopiped.

Each distinct variety of parallelopiped, not only produces a distinct variety of tetrahedron, octahedron, dodecahedron and trapezohedron; but likewise admits of indefinite variations: consequently an indefinite number of parallel series may be formed, each of which series will be regulated by the same general law. These variations will be found to embrace the greatest part, if not the whole of the crystaline forms that have been discovered among minerals.

Having investigated the various properties of the Platonic bodies, and those solids which may be derived from them, by removing their different extremities in a direct manner; and having pointed out the important fact, that these solids and their derivatives naturally divide themselves into two series; that each individual in one of the series is to be found in great abundance among the productions of nature, in the mineral department; and that not a single individual in the other series has ever been found among such or any other of nature's productions; the next object will be to describe those angular geometrical solids that rank next in importance. These are the rhombohedron; five varieties of dodecahedrons; the prisms, the pyramids, and the parallelopipeds.

THE RHOMBOHEDRON, OR RHOMBOID.

This solid has

Six faces,
Eight angular extremities,
Twelve edges,
Twenty-four plane angles, and
One axis.

Its faces are equal and similar. Its edges are of two kinds, one half may be called its superior or inferior, and the other half its lateral edges. Its angular extremities are of two kinds, two of which may be called its summits and the remaining six its lateral extremities.

In the acute rhombohedron, (*fig.* 95.) the summits are formed of three acute plane angles; and the lateral solid angles are each formed by the meeting of two obtuse and one acute plane angle. The lines uniting the acute plane angles are the oblique diagonals, and the lines uniting the obtuse plane angles are the horizontal diagonals.

In the obtuse rhombohedron (*fig.* 96), the summits are formed by three obtuse plane angles meeting together, and the lateral solid angles are formed, by the meeting of one obtuse and two acute plane angles, the lines uniting its obtuse plane angles are its oblique diagonals, and the lines uniting its acute plane angles are its horizontal diagonals.

The rhombohedron is a parallelopiped, which is generated from a cube, by elongating or shorting one of its axis; if the axis of the cube be elongated, it will become an acute rhombohedron, but if it be shortened it will become an obtuse rhombohedron. On account of its six faces being equal and similar through all its variations, it may be called a hexahedron: in that case, the acute variety may be called an acute hexahedron, the cube a right angled hexahedron, and the obtuse variety an obtuse hexahedron.

Each variety admits of indefinite variations. The obtuse proceeds from the cube by shortening one of its axes; when that axis disappears the six faces collapse together, forming a plane, whose boundaries is a regular hexagon. The acute variety proceeds from a cube by elongating one of its axes, till that axis becomes a line of indefinite extent, when the solid disappears in that line. So that at one extremity of its variations it terminates in *a line*, at the other in *a plane*, and in the intermediary state it is *a solid.*

A rhomboidal dodecahedron may be divided into four obtuse rhombohedrons, like *fig.* 100. If an octahedron have a tetrahedron applied to each of two opposite faces, it will be converted into an acute rhombohedron, like *fig* 99. These two rhombohedrons have this curious property, that if a perpendicular be drawn from the angular extremity of an obtuse plane angle of either of the solids, and a plane pass through it, cutting the solids at right angles to the face on which the perpendicular is drawn; the face of the section of either of the solids, will be similar to the external face of the other solid.

Of the five varieties of dodecahedrons to be described, three of them have triangular faces, one has quadrangular faces, and one has pentagonal faces. One of the dodecahedrons with triangular faces is bounded by scalene triangles, and may be called a scalene dodecahedron; a second has the appearance of two hexangular pyramids joined base to base, and has been called from that circumstance, a bipyramidal dodecahedron; a third is formed of isosceles triangles, whose vertical angle is always more than double of either of the angles at the base, and may be called a tetradodecahedron. The dodecahedron with quadrangular faces, which is bounded by trapezoidal faces similar to the faces of the trapezohedren or hexacontahedron, may be called the trapezoidal dodecahedron. The dodecahedron bounded by pentagonal faces, which are not regular pentagons, but are all similar, may be called a pentagonal dodecahedron.

SCALENE DODECAHEDRON,

(FIG. 95.)

This solid has

Twelve faces,
Eight solid angles,
Eighteen edges,
Thirty-six plane angles,
And one axis.

Its faces are equal and similar. Its solid angles are of two varieties; two of them may be called its summits, the others its lateral angles. It has two varieties of edges; twelve may be called its superior edges, and the remaining six its lateral edges. Its plane angles are of three varieties, twelve obtuse, twelve acute, and the remaining twelve still more acute.

The lateral edges coincide with the edges of a contained rhombohedron.

This variety of dodecahedron, admits of more variations than any other; independently of the variations produced by elongating and shortening its axis, an indefinite variety may be formed by varying the inclination of the lateral edges to each other, which may proceed from a very acute angle (*fig.* 95,) to a more obtuse angle, (*fig.* 96,) and when that angle becomes 120° all the lateral edges will be in the same plane (*fig.* 97.) This plane will divide the solid into two hexagonal pyramids, on which account it may then be called a bipyramidal dodecahedron, which will form the next variety.

BIPYRAMIDAL DODECAHEDRON.

(*FIG.* 97.)

This solid has the same number of faces, edges and angles as the scalene dodecahedron: it has four axes.

Its faces are all isosceles triangles, consequently there are but two varieties of plane angles; twelve of which must always be less than 60°, and the remaining twenty-four must be each greater than 60°.

There is one variety which may be generated by the intersection of two cubes; all the remaining variations may be made by the intersections of two rhombohedrons. In each of these cases the axes of the two intersecting solids must coincide, and their lateral edges must bisect each other; when the superior and inferior edges of the one will be exactly over the oblique diagonals of the other.

THE TETRADODECAHEDRON.

(FIG. 92.)

This solid has

Twelve faces,
Eight solid angles,
Eighteen edges,
Thirty-six plane angles, and
Four axes.

Its faces are all equal and similar isosceles triangles. It has two varieties of solid angles, four acute and four obtuse; the acute are each formed by the meeting of six acute plane angles, and the obtuse by the meeting of three obtuse plane angles. It has six long edges and twelve short. It has two varieties of plane angles, twelve obtuse, which are never greater than 120°; and twenty-four acute, which always exceed 30°, but never equal 45°. It has four axes which are all equal: each of these axes unites an acute angular extremity with an obtuse angular extremity, which are opposite to each other.

It admits of considerable variations; it has the appearance of a tetrahedron, with an obtuse triangular pyramid on each of its faces. When the obtuse plane angles are equal to 120°, the pyramids disappear, and it becomes a tetrahedron; when the obtuse angle becomes a right angle the solid is then converted into a cube.

TRAPEZOIDAL DODECAHEDRON.

(FIGS. 93 & 94.)

This solid has

Twelve faces,
Fourteen angular extremities,
Twenty-four edges,
Forty-eight plane angles, and
Seven axes.

Its faces are all equal. It has three varieties of solid angles, four obtuse, four acute, and six intermediary. Its edges are of two kinds, twelve long and twelve short. It has three varieties of plane angles, twelve obtuse, which never equal 120°, twenty-four acute, that vary between 90° and 70° 31′ 43″½, and twelve that are sometimes obtuse and sometimes acute, varying between 109° 28′ 16″½ and 60°. It has two kinds of axes; when it approaches the rhomboidal dodecahedron it has three long and four short axes, but when it approaches the tetrahedron, it has four long and three short; there is one stage in the variations when they are all equal.

This is the only solid that can be made of twelve equal and similar trapezoidal faces, grouped together, in a similar manner as the trapezohedron and the hexacontahedron *(see demonstrations:)* it passes into the tetrahedron and the rhomboidal-dodecahedron; in *fig.* 93 it is approaching the tetrahedron, and in *fig.* 94 it is approaching the rhomboidal dodecahedron. Its acute angu lar extremities coincide with the angular extremities of the contained tetrahedron, its obtuse angular extremities with the angular extremities of the contained cube, and its intermediary angular extremities, with the angular extremities of the contained octahedron, whose edges coincide with one half of the diagonals, which unite its intermediary angular extremities, as *a*, *b*.

If the face *A b d a*, move upon the diagonal *a b*, as upon an axis, till the opposite angles *a A b* and *a d b* are equal, and all the faces change their position in a similar and equal manner, the solid will become a rhomboidal dodecahedron: if the obtuse angular extremity *d*, be so depressed that it coincide with the faces of the contained tetrahedron, the trapezoidal faces will disappear, and the solid will pass into that tetrahedron.

THE PENTAGONAL DODECAHEDRON.

(*FIG.* 90.)

This solid has

Twelve faces,
Thirty edges,
Twenty angular extremities,
Sixty plane angles, and
Four axes.

Its faces are all equal and similar pentagons united together in pairs. Its solid angles are of two varieties, eight of which are formed by the meeting of three equal plane angles, and the remaining twelve are formed by the meeting of unequal plane angles. It has two varieties of edges, in one set of variations six of them are longer than the remaining twenty-four, in the other set of variations they are shorter.

Each face of this variety of dodecahedron, has four of its edges always equal; the fifth edge (*fig.* 91) is either longer or shorter than the rest; the opposite angle belonging to the same face is more obtuse than the other angles; and when that edge is equal to the diagonal, which is parallel to it on the same face, the opposite angle disappears, and the edges forming it become that diagonal. A similar change being supposed to take place with all the faces, the solid will then become a cube. But should the above mentioned edge be shorter than the rest, as *fig*. 90, the opposite angle becomes more acute than the rest, and continues to become more so, as the edge becomes shortened. In all the changes that have been mentioned, the four remaining edges are in every stage equal, and continue so when the above line disappears, in which case they unite together and form a rhomb, and the solid becomes a rhomboidal dodecahedron. Hence a pentagonal dodecahedron, at one extremity of the series of its variations becomes a cube, and at the other end a rhomboidal dodecahedron.

Seven distinct varieties of dodecahedrons have been described; the regular dodecahedron, the rhomboidal dodecahedron, the scalene dodecahedron, the bipyramidal dodecahedron, the tetradodecahedron, the pentagonal dodecahedron, and the trapezoidal dodecahedron. The trapezoidal dodecahedron passes insensibly into the rhomboidal dodecahedron, *see figs*. 93 *and* 94; this passes imperceptibly into the pentagonal dodecahedron, *see* *fig.* 90 *and* 91, which passes into a cube at the other extremity of its series of variations; but becomes in one of its stages a regular dodecahedron. The pentagonal dodecahedron having disappeared in the cube, the tetradodecahedron imperceptibly proceeds from it, by drawing the diagonals which coincide with the edges of one of the contained tetrahedrons, and at the same time depressing the angular extremities of the cube that are opposite the faces of the above tetrahedron. If a cube be placed on one of its angular extremities, as *fig.* 36, and its six oblique diagonals be drawn, they will shew how the scalene dodecahedron may arise from the cube; it was before shown that the scalene dodecahedron may insensibly become a bipyramidal dodecahedron. Hence these seven dodecahedrons are most intimately united by means of the cube, and they may all be said to proceed insensibly from that solid.

THE PRISM

Is a solid that admits of great variations; it is in general distinguished by the number of sides bounding its base. It may be distinguished into right or oblique; a right prism has its sides at right angles to its base, an oblique prism has none of its sides at right angles to its base, except in some few instances in which two of its sides may be at right angles.

They are named from the number of sides which bound their bases. Hence a prism with a triangular base, is called a triangular prism; with a quadrangular base, a quadrangular prism, &c.

Triangular, quadrangular, hexangular and octangular prisms, are to be found in great abundance among the productions of nature. The hexagonal prism is considered by some, to be generated by the intersection of two triangular prisms; the octagonal prism by the intersection of two quadrangular prisms.

THE PYRAMID.

This solid like the prism is distinguished by the number of sides that bound its base; and likewise by its being right or oblique; when its summit is directly over the centre of its base, it is a right pyramid; when it is not it is an oblique pyramid.

All pyramids which have an equal base and altitude with a prism are equal to one-third of that prism, whether they be right or oblique.

Triangular, quadrangular, and hexangular pyramids are found in abundance among crystals.

THE PARALLELOPIPED.

This solid admits of great variations; the cube and the two rhombohedrons are three of them; the remaining varieties are the right and oblique quadrangular prisms: if these are distinguished by their bases, they will each admit of four distinct varieties, namely, a right quadrangular prism with a square base, (*fig.* 81;) a right quadrangular prism with a rectangular base, (*fig.* 82;) a right quadrangular prism with a rhombic base, or whose base is a rhomb, (*fig.* 83;) a right quadrangular prism with a rhomboidal base, or whose base is an oblique angled parallelogram, (*fig.* 84.) The oblique varieties may be distinguished in a similar manner.

The oblique prisms may be distinguished in another manner into three varieties, namely, oblique on the edge, oblique on the angle, and the awry prism, which partakes of both. This division of oblique prisms has been introduced by Dr. Wollaston.

There are other angular solids, some of which may perhaps, not be improperly termed fanciful solids; these are formed in a variety of ways, but chiefly by removing the extremities of those solids already described in any way the fancy may lead, without any direct rule or law like what has been employed in the series already mentioned. A few remarkable solids of this kind may be seen in Sharp's Geometry Improved. There are a few solids that have not been described in the present work, such as the wedge, the prismoid, &c. useful in the study of mensuration; these may be seen described in books on that subject.

THE CONE, SPHERE, AND CYLINDER.

When the diameter of the base of a cone with its altitude; the diameter of the sphere, and the diameter of the base of a cylinder with its altitude, are all equal, their volumes are to each other as 1, 2 and 3.

The sections of a sphere are all circles.

The sections of a cylinder parallel to its base are circles; all sections that are not parallel to the base or axis are ellipses; but all sections that are parallel to its axis are parallelograms.

The sections of a cone, parallel to its base, are circles. The sections of a cone parallel to its side are parabolas; all sections between these two. or all sections which make a more obtuse angle with its axis than the sides, are ellipses; but all sections that make a more acute angle with the axis, or are parallel with it, are hyperbolas; all sections passing through the summit and its axis are triangles. There have been a number of treatises written on these sections describing their properties.

DEMONSTRATIONS

OF A NUMBER OF

The Properties,

BELONGING TO THE

Natural and Artificial Series.

DEMONSTRATIONS

OF A VARIETY OF PROPERTIES BELONGING TO THE

NATURAL SERIES.

DEFINITIONS.

THE relative positions of the different solids belonging to the natural series, when contained in or enveloped by each other, are determined by their situation in the trapezohedron.

The terms contained and enveloped, are used in some instances, instead of the terms inscribed and circumscribed: For in many cases the position of the contained solid, does not agree with the definition of an inscribed solid, which is, "that the angular extremities of the inscribed solid, terminate in the centres of the faces of the circumscribed solid," nor is it possible for some to be so placed. The different positions of the contained solids may be divided into three classes.

The First, When the angular extremities, of the contained solid, coincide with certain sets of the angular extremities of the enveloping solid.

The Second, When certain faces of the contained solid coincide, or are in the same plane, with the faces of the enveloping solid. In this case, the angular extremities of the contained solid bisects all the edges of the enveloping solid, with the exception of one instance, when, instead of bisecting the edges, it bisects all the perpendiculars of the faces.

The Third, When the angular extremities of the contained solid terminate in the centres of the faces of the enveloping solid.

The two first classes will be distinguished by the terms contained and enveloped; and the third, by the terms inscribed and circumscribed.

THE TRAPEZOHEDRON

Of the relative position of the natural solids, when contained in the trapezohedron.

1. *The Rhomboidal Dodecahedron in the Trapezohedron.*

The acute angular extremities, of the contained rhomboidal dodecahedron, terminate in the acute angular extremities of the trapezohedron: and the obtuse angular extremities of the one, in the obtuse angular extremities of the other. The edges of the rhomboidal dodecahedron coincide with the short diagonals of the trapezohedron.

2. *The Octahedron in the Trapezohedron.*

The angular extremities of the contained octahedron, terminate in the acute angular extremities of the trapezohedron.

3. *The Cube in the Trapezohedron.*

The angular extremities of the contained cube terminate in the obtuse angular extremities of the trapezohedron.

4. *The Tetrahedron in the Trapezohedron.*

The angular extremities, of the contained tetrahedron, terminate in the alternate obtuse angular extremities of the trapezohedron.

5. *The Cuboctahedron in the Trapezohedron.*

The angular extremities, of the contained cuboctahedron, terminate in the intermediary angular extremities of the trapezohedron; and its edges coincide with the long diagonals.

THE RHOMBOIDAL DODECAHEDRON.

Of the relative position of the natural solids, when contained in the rhomboidal dodecahedron.

6. *The Octahedron in the Rhomboidal Dodecahedron.*

The angular extremities of the contained octahedron terminate in the acute angular extremities, of the rhomboidal dodecahedron; and its edges coincide with the long diagonals.

7. *The Cube in the Rhomboidal Dodecahedron.*

The angular extremities of the contained cube terminate in the obtuse angular extremities of the rhomboidal dodecahedron; and its edges coincide with the short diagonals.

8. *The Tetrahedron in the Rhomboidal Dodecahedron.*

The angular extremities, of the contained tetrahedron, terminate in the alternate obtuse angular extremities of the rhomboidal dodecahedron.

9. *The Trapezohedron in the Rhomboidal Dodecahedron.*

The intermediary angular extremities, of the inscribed trapezohedron, terminate in the centres of the faces of the rhomboidal dodecahedron.

10. *The Cuboctahedron in the Rhomboidal Dodecahedron.*

The angular extremities, of the inscribed cuboctahedron, terminate in the centres of the faces of the rhomboidal dodecahedron.

THE OCTAHEDRON.

Of the relative positions of the natural solids, when contained in an octahedron.

11. *The Cube in an Octahedron.*

The angular extremities of an inscribed cube terminate in the centres of the faces of an octahedron.

12. *The Tetrahedron in the Octahedron.*

The angular extremities of the inscribed tetrahedron terminate in the centres of the alternate faces of the octahedron.

13. *The Trapezohedron in the Octahedron.*

The obtuse angular extremities of the inscribed trapezohedron terminate in the centres of the faces of the octahedron.

14. *The Rhomboidal Dodecahedron in the Octahedron.*

The obtuse angular extremities, of the inscribed rhomboidal dodecahedron, terminate in the centres of the faces of the octahedron.

15. *The Cubootahedron in the Octahedron.*

The angular extremities of the contained cubootahedron terminate in, and bisect the edges of the octahedron; and its triangular faces coincide with the faces of the octahedron.

THE CUBE.

Of the relative positions of the natural solids, when contained in a cube.

16. *The Tetrahedron in a Cube.*

The angular extremities of the contained tetrahedron terminate in the alternate angular extremities of the cube; and its edges coincide with the diagonals of the cube.

17. *The Trapezohedron in a Cube.*

The acute angular extremities, of the inscribed trapezohedron terminate in the centres of the faces of the cube.

18. *The Rhomboidal Dodecahedron in a Cube.*

The acute angular extremities, of the inscribed rhomboidal dodecahedron, terminate in the centres of the faces of the cube.

19. *The Octahedron in a Cube.*

The angular extremities of the inscribed octahedron terminate in the centres of the faces of the cube.

20. *The Cuboctahedron in a Cube,*

The angular extremities of a contained cuboctahedron terminate in, and bisect the edges of the cube; and its square faces coincide with the faces of the cube.

THE TETRAHEDRON.

Of the relative position of the natural solids, when contained in a tetrahedron.

21. *The Trapezohedron in a Tetrahedron.*

The alternate obtuse angular extremities, of an inscribed trapezohedron, terminate in the centres of the faces of the tetrahedron.

22. *The Rhomboidal Dodecahedron in a Tetrahedron.*

The alternate obtuse angular extremities, of an inscribed rhomboidal dodecahedron, terminate in the centres of the faces of the tetrahedron.

23. *The Octahedron in a Tetrahedron.*

The angular extremities of the contained octahedron terminate in and bisect the edges of the tetrahedron; and its faces coincide with the faces of the tetrahedron.

24. *The Cube in a Tetrahedron.*

The alternate angular extremities of the inscribed cube terminate in the centres of the faces of the tetrahedron.

25. *The Cuboctahedron in a Tetrahedron.*

The angular extremities of the contained cuboctahedron terminate in, and bisect all the perpendiculars on the faces of the tetrahedron; and its triangular faces coincide with the faces of the tetrahedron.

THE CUBOCTAHEDRON.

Of the relative position of the natural solids, when contained in a cuboctahedron.

26. *The Trapezohedron in a Cuboctahedron.*

The acute angular extremities of the inscribed trapezohedron terminate in the centres of the square faces of the cuboctahedron.

27. *The Rhomboidal Dodecahedron in a Cuboctahedron.*

The acute angular extremities, of the inscribed rhomboidal dodecahedron, terminate in the centres of the square faces of the cuboctahedron.

28. *The Octahedron in a Cuboctahedron.*

The angular extremities of the inscribed octahedron terminate in the centres of the square faces of the cuboctahedron.

29. *The Cube in a Cuboctahedron.*

The angular extremities of the inscribed cube terminate in the centres of the triangular faces of the cuboctahedron.

30. *The Tetrahedron in a Cuboctahedron.*

The angular extremities of a tetrahedron terminate in the centres of the alternate triangular faces of the cuboctahedron.

THEOREM I.

The perpendicular of an equilateral triangle is divided by the centre into two parts, that are to each other as 2 : 1.

In the triangle *ABD* (*fig.* 62,) the line *An* passes through the centre *e*, which centre cuts off *en* equal to half *Ae*. Because *eDc* and *eBC* are two equal, equilateral triangles, and the base *eC*, common to both, the line *DB* will bisect *eC* in *n*, making *en* equal to half *eC*, equal to half *Ae*. *Therefore, the perpendicular, &c.*

THEOREM II.

The two Diagonals of the face of a Rhomboidal Dodecahedron, are to each other, as $\sqrt{2} : 1$.

AB and *BC* (*fig.* 34,) being the short diagonals of the rhomboidal dodecahedron, they will likewise be the edges of the contained cube, which are consequently at right angles to each other; therefore *AC* the diagonal of the cube will be to either of them as $\sqrt{2} : 1$. But *AC* is equal to *EF*, one of the long diagonals; hence *EF*, which is equal to *AC*, is to *BC*, one of the short diagonals, as $\sqrt{2} : 1$. *Therefore, the two diagonals, &c.*

THEOREM III.

The inclination of the faces of an Octahedron, are equal to the plane angles of a rhomboidal dodecahedron.

If the perpendiculars *Ac*, *cB*, *Bd* and *dA* (*fig.* 42) be drawn, they will be in the same plane; which plane will divide the octahedron into two equal parts, and the face of the section will be a rhomb, whose diagonals will be *cd* and *AB*, which last is the axis of the octahedron. *ADBC* is a square of which *AB* is a diagonal; therefore *AB* will be to *BC*, as the $\sqrt{2}:1$; but *BC* is equal to *CF*, equal to *dc*; consequently *AB* is to *dc*, as $\sqrt{2}:1$. Hence *AcBd* will be similar to the face of a rhomboidal dodecahedron[a], and because *AcB* is the obtuse, and *cAd* the acute inclinations[a] of the faces of an octahedron. *Therefore, the inclinations, &c.*

THEOREM IV.

If through the centre of the face of a Rhomboidal Dodecahedron, a line be drawn at right angles to one of its edges; and from an obtuse angular extremity of the face another line be drawn parallel to the first mentioned line, these two lines will divide the edge into three equal parts.

Let the section of the octahedron, *fig.* 42. be represented by *ABCD* (*fig.* 63,) *a*, the centre of the upper face, of which *AD* is the perpendicular, therefore *Aa* will be equal to $\frac{1}{3}$ *AD*[b]; *d*, the centre of the lower face, of which *CB* is the perpendicular; therefore *Cd* is equal to $\frac{1}{3}$ *CB*,[b] equal *aA*; *ad*, which passes through the centre *b*, is the altitude of the octahedron, and consequently is perpendicular to *AD* and *CB*; from *A*, draw *Ac* parallel to *ad*, *aA* being equal to *Cd*; *cd* will also be equal to *dC*, equal to $\frac{1}{3}$ *CB*, thence *cB* will likewise be equal to $\frac{1}{3}$ *CB*. *Therefore, if through the centre, &c.*

THEOREM V.

The edge of the Cuboctahedron is equal to half the diagonal of the cube from whence the Cuboctahedron is formed.

If all the edges of a cube (*fig.* 80) be bisected, and the adjacent points of bisection be united by lines, those lines will coincide with the edges of the contained cuboctahedron; if in one of the faces *ABCD* of a cube, the opposite points of bisection, *a*, *c* and *b*, *d* be united by lines, those lines will be parallel to the sides of the face, and will divide it into four equal squares, of which the edges of the contained cuboctahedron will be the diagonals; but the diagonals of each of the squares is equal to half the diagonal *AC* of the face of the cube. *Therefore, the edges, &c.*

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[a] Theo. II. [b] Theo. I.

THEOREM VI.

Lines drawn from the angular extremities of a Tetrahedron to to its centre, meet at angles equal to the obtuse plane angles, of the faces of a rhomboidal dodecahedron.

Because lines drawn from the angular extremities of a tetrahedron, terminating in the centre of the solid; divide it into four equal and similar triangular pyramids; therefore the inclination of the sides of the pyramids, on their respective bases, will be equal to half the inclination of the faces of the tetrahedron; equal to half the supplement of the obtuse inclination of the faces of an octahedron; or half the difference of that inclination and two right angles. Hence if two of these pyramids be applied, to the two adjacent faces of an octahedron; the two sides of the pyramids which are united together, by the edges of the octahedron, will be in the same plane. If one of these pyramids be placed on each of the faces of the octahedron, there will be twelve such faces formed, one on each edge of the octahedron, making a dodecahedron; but because no other dodecahedron, than the rhomboidal dodecahedron can be formed on an octahedron; so that the faces of the dodecahedron, shall have a similar relation to the edges of the octahedron; therefore the summits of the triangular pyramids will be similar, to the obtuse solid angles of a rhomboidal dodecahedron; consequently the obtuse plane angles of the one will be equal to the obtuse plane angles of the other; but the obtuse plane angles of the triangular pyramids are formed, by the meeting of lines drawn from the angular extremities of a tetrahedron, terminating in its centre. *Therefore, lines drawn, &c.*

Cor. Hence a rhomboidal dodecahedron is equal to its contained octahedron and two tetrahedrons, whose faces are equal to the faces of the octahedron.

THEOREM VII.

Lines drawn from the angular extremities of a cube, terminating in the centre; meet together at angles, equal to the obtuse and acute plane angles, of a rhomboidal dodecahedron.

Because lines drawn from the angular extremities of a cube, terminating in the centre, divide the cube into six equal and similar quadrangular pyramids; therefore the inclination of the sides of these pyramids on their bases will be equal to half the inclination of the faces of the cube, equal to half a right angle; if two of these pyramids be placed on two adjacent faces of a cube, whose faces are equal to the bases of the pyramids; the two sides of the pyramids, united by the edge of the cube, will form a continued plane on that edge; and if similar pyramids be applied to the remaining faces of the cube, they will consequently form together a rhomboidal dodecahedron; therefore the summits of the pyramids will be similar to the acute solid angles of a rhomboidal dodecahedron *(fig* 64,*)* whose edges *AF*, *Fi* and *FE*, bounding the acute solid angle *F*, have the same inclination between themselves, as the obtuse and acute plane angles of one of the faces, because *AF* is parallel to *mi*, and *FE* is parallel to *io*, therefore the angle *AFE* is equal to the angle *mio*, one of the obtuse plane angles of the faces; and *AFi* is one of the acute plane angles. *Therefore, lines drawn, &c.*

Cor. Hence the volume of the six pyramids will be equal to the volume of the contained cube.

THEOREM VIII.

Lines drawn from the angular extremities of an Octahedron, terminating in its centre will meet at right angles.

Because lines drawn, from the angular extremities of an octahedron terminating in its centre, divide the octahedron into eight equal and similar triangular pyramids; therefore the inclinations of their sides on their bases will be equal to half the inclination of the faces of the octahedron, or half the supplement of the inclination of the faces of a tetrahedron: if two of these pyramids be applied to the faces of a tetrahedron; the sides of the pyramids, united by the same edge of the tetrahedron, will form a continued plane on that edge; consequently this plane will coincide with the face of an enveloping cube. If similar pyramids be applied to each face of the tetrahedron, they will form a complete cube; one half of the angular extremities of the cube, being formed by the summits of the pyramids; therefore the summits of the pyramids will be similar to the solid angles of the cube; consequently the plane angles of the summits will be right angles; and the edges forming them will also meet at right angles. *Therefore, lines drawn, &c.*

THEOREM IX.

AXIS OF THE CUBE.

A plane passing through three diagonals of a Cube, cuts off one third from its axis.

From *d* and *F* (*fig.* 36), let fall the perpendiculars *da* and *Fb*, on the axis *AB*; because the angles *FbA* and *daA* are both right angles, and the angle FAb is common to both: therefore the triangles FAb and dAa are equi-angular, and $Aa : Ab :: Ad : AF$: but Ad is equal to half AF, therefore Aa is equal to ab. Because C, D and E are equally distant from the axis AB, and from A, the extremity of that axis; therefore the axis is perpendicular to the plane passing through these three points; and because the point d is in that plane, and da is at right angles to the axis AB; therefore da is in the plane passing through C, D and E, which plane cuts the axis AB, in a; because Aa is perpendicular to the plane passing through C, D and E; therefore all lines drawn in that plane, terminating in a, will be at right angles to the axis AB; hence Ea will be at right angles to AB. In a similar manner it may be demonstrated, that cb is at right angles to AB, and consequently the triangles Bbc and BaE, will be right angled triangles, and having the angle EBa, common to both, the triangles will be equi-angular; and $Bb : Ba :: Bc : BE$; but Bc is equal to half BE, therefore Bb is equal to half Ba, equal to ba; but Aa was proved to be equal to ba, therefore Aa is equal to Bb; consequently the axis is divided into three equal parts by the points a and b. *Therefore, a plane, &c.*

THEOREM X.

THE SHORT AXIS OF THE RHOMBOIDAL DODECAHEDRON.

If a plane pass through the acute solid angles of a Rhomboidal Dodecahedron, at right angles to its shortest axis; it will cut off one sixth from that axis; and passing through the obtuse angular extremities, it will cut off one third; but passing through the centre of the face, it cuts off one half, or bisects that axis.

On the face of the rhomboidal dodecahedron (*fig.* 64,) let the perpendicular gl be drawn through the centre c; and ih

and sn be drawn through i and s parallel to gl; then will the edges, mn and io be divided into three equal parts[c]; and a plane, passing through gl at right angles to the axis AD, will bisect that axis in c; and a plane passing through n at right angles to the same axis, will cut it in d, making cd equal to ln; if another plane pass through ih, in the same direction it will cut AD in b, making bc equal to hl; hence bc and cd will be each equal to one third the edge mn. A plane passing through the acute angular extremities F and m, at right angles to the axis AD will pass through the long diagonal Fm, and bisect the short diagonal Ai in t, and because the eye is supposed to be, at an infinite distance from (*fig.* 64;) therefore planes passing through the lines Fm, ih, *&c.* at right angles to the axis AD, will disappear in these lines; together with all lines drawn in the same plane; as the perpendiculars ta and ma, drawn from t and m on the axis AD, which perpendiculars coincide with, or disappear in the line Fm; and the perpendiculars ib and hb drawn from i and h, which coincide with, or disappear in the line ih. Hence the triangles, Ata and Aib, have the angles Aat and Abi both right angles, and the angle iAb common; therefore $Aa : Ab :: At : Ai$, but At is equal to $\frac{1}{2} Ai$; consequently Aa will be equal to half $Ab = ab$; and because the perpendiculars, ma and hb, are parallel; therefore $ab = mh$. Hence Aa and ab are each equal to one third of the edge of the rhomboidal dodecahedron. In a similar manner, it may be demonstrated, that De and ed are each equal to one third the same edge: therefore the axis AD is divided into six equal parts. Hence a plane passing through the obtuse angular extremities i and B, at right angles to the short axis AD, will cut off Ab, equal to one third of that axis; and a plane, passing through the acute angular extremities F and m, at right angles to the same axis, cuts off one sixth from that axis. *Therefore, if a plane, &c.*

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[c] Theo. III.

Cor. Hence the edge of a rhomboidal dodecahedron is equal, to half the shortest axis.

THEOREM XI.

THE LONGEST AXIS OF A RHOMBOIDAL DODECAHEDRON.

If a plane pass through the acute angular extremities of a Rhomboidal Dodecahedron, at right angles to one of the longest axes, it will bisect that axis; but if it pass through the obtuse angular extremities, it will cut off one fourth from the same axis.

If a plane pass through the acute angular extremities of a rhomboidal dodecahedron, at right angles to the longest axis, it will likewise pass through four long diagonals, and divide the solid into two equal parts, and bisect that axis. But if it pass through the obtuse angular extremities, at right angles to the same axis, it will at the same time pass through four short diagonals, and cut off a quadrangular pyramid, whose altitude will be equal to that part of the axis which is cut off by the plane; equal to half the altitude of the contained cube; but the altitude of the contained cube, together with the altitude of two quadrangular pyramids, situated on two of its opposite faces, are equal to one of the longest axes; and the altitude of either of the pyramids being equal to half the altitude of the cube; consequently the altitude of one of the quadrangular pyramids is equal to one fourth of the longest axis. *Therefore, if a plane, &c.*

THEOREM XII.

The long diagonal of the face of a Trapezohedron cuts off one third from the short diagonal, or trisects it.

Because a plane passing through the centre of the faces of a contained rhomboidal dodecahedron, at right angles to its shortest axis cuts off one third from the edges which are parallel to that axis[d]; and because one of the edges of the contained rhomboidal dodecahedron coincides with the short diagonal *gp*, of the trapezohedron (*fig.* 66); and the above section passes through the long diagonal *cd*, that crosses the short diagonal *gp*. *Therefore, the long diagonal, &c.*

THEOREM XIII.

THE SHORTEST AXIS OF THE TRAPEZOHEDRON.

If a plane cut a Trapezohedron at right angles to its shortest axis, and pass through the obtuse angular extremities in the same direction, it cuts off one third; and through the acute angular extremities, it cuts off one sixth; but through the intermediary angular extremities, it cuts off one eighteenth.

BC (*fig.* 66,) which is the shortest axis of the trapezohedron, is likewise the shortest axis of the contained rhomboidal dodecahedron: because the obtuse angular extremities of the rhomboidal dodecahedron coincide with the obtuse angular extremities *B*, *g* and *c*, &c. of the trapezohedron; and likewise the acute angular extremities of the rhomboidal dodecahedron coincide in a similar manner with the acute angular extremities *Aip*, *&c*; consequently whatever plane passes through the acute or obtuse angular extremities of the one, passes through the acute or obtuse angular extremities of the other, and cuts the axis in a similar manner; therefore a plane passing through the obtuse angular extremity *g*, at right angles to the axis *BC*, cuts of one third from that axis[e]; and passing through the acute angular extremities *A* and *i* cuts off one sixth from the same axis[e].

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[d] Theo. X.

To determine what portion of the shortest axis is cut off, by the plane passing through the intermediary solid angles *n* and *b*; draw the triangle *Bai* (*fig.* 65), whose side *Bi* is equal to the short diagonal of a trapezohedron, and whose side *Ba* is equal to one sixth of the shortest axis, with an angle *aBi*, equal to the inclination of the short diagonal of the face on the shortest axis; join *a* and *i*, then will *ai* be at right angles to *aB*. Because the long diagonal of the face of a trapezohedron cuts off one third from the short diagonal, make *Be* equal to one third *Bi*, and draw *es* parallel to *ia*; then will *Be* : *Bi* :: *Bs* : *Ba*; as *Be* is one third *Bi*, therefore *Bs* is one third *Ba*; but *Ba* is equal to one sixth of the axis *BC* (*fig.* 66 :) consequently *Bs* (*fig.* 65) is equal to one eighteenth of *BC* (*fig.* 66;) which is the part cut off, by a plane passing through the intermediary angular extremities *n* and *b*, at right angles to the shortest axis *BC*. *Therefore, if a plane, &c.*

Cor. Because the short diagonal of the trapezohedron is equal to the edge of the contained rhomboidal dodecahedron, it is equal to half the shortest axis.

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[e] Theo. X.

THEOREM XIV.

THE MIDDLE AXIS OF THE TRAPEZOHEDRON.

If a plane pass through the short diagonal of a face of a Trapezohedron, at right angles to that face, it will bisect the middle axis; if another plane pass through the intermediary angular extremities, at right angles to the same axis, it will cut off one fourth; and if a plane pass through the obtuse and acute angular extremities next above, it will cut off one eighth from that axis.

The trapezohedron (*fig.* 67) is so placed, that one of its obtuse angular extremities *I* occupies the centre of the figure; *lNi* are three intermediary angular extremities; *H*, *a* and *e* are three acute angular extremities; *G*, *h* and *b* are three obtuse angular extremities; six of the faces that bound the solid, disappear in the lines *AB*, *BC*, *CD*, *DE*, *EF* and *FA*, which lines are equal to the long diagonals of those faces.

The long diagonals *BC*, *Ni* and *FE*, being parallel to the middle axis *AD*; they will be perpendicular to the plane, that cuts the axis *AD* at right angles through the extremity *I*, which plane will pass through the extremities *a*, *I* and *b*: and because *Ba* and *Fb* are are each equal to *Nd*, half the long diagonal *Ni*; therefore the intermediary angular extremities, *B*, *N* and *F*, will be of the same altitude above that plane; hence a plane passing through them will be parallel to it, and will consequently cut the axis *AD* at right angles. Because *ABCDEF* is a regular hexagon, therefore *FB* will bisect *AI*, making *An* equal to half *AI*, equal $\frac{1}{4}$ *AD*; consequently a plane, passing through the intermediary angular extremities *F*, *N* and *B* will cut off *An* equal to $\frac{1}{4}$ *AD*; and because the acute angular extremity *H* bisects *AF*, and the obtuse angular extremity G bisects AB; therefore HG will be parallel to FB, and a plane passing through HG at right angles to AD will bisect An making Ar equal to half An equal to $\frac{1}{8}$ AD. Therefore a plane passing through the obtuse angular extremity G, or the acute angular extremity H, at right angles to AD, will cut off one eighth from the axis AD.

Cor. I. Because Nd, which is equal to half Ni, is equal to nI, half AI; therefore the diagonal Ni is equal to half the middle axis AD.

Cor. II. Hence either of the diagonals of the face of a trapezohedron is equal to half the axis that is parallel to it.

THEOREM XV.

THE LONGEST AXIS OF THE TRAPEZOHEDRON.

If a plane pass through the acute angular extremity of a Trapezohedron, at right angles to the longest axis; it will bisect the axis: if it pass through an obtuse angular extremity, in the same direction, it will cut off one fourth; but if it pass through the intermediary angular extremities next above, it will cut off one sixth.

Because the longest axis, of the contained rhomboidal dodecahedron coincides with the longest axis of the trapezohedron; and likewise the obtuse and acute angular extremities of the one, with the obtuse and acute angular extremities of the other; and because a plane passing through the obtuse angular extremities of a rhomboidal dodecahedron, at right angles to the longest axis cuts off one fourth from that axis; therefore a plane passing through the obtuse angular extremities of a trapezohedron, at right angles to its longest axis, will likewise cut off one fourth from that axis.

A, *D*, *B*, *C* and *g* (*fig.* 68) are the acute angular extremities, of the trapezohedron; *h*, *a*, *c* and *e*, the intermediary angular extremities; and *d*, an obtuse angular extremity; *AB* is one of the longest axis; *dn* is a perpendicular let fall from the point *d*, on *AB*; and *ba* is a perpendicular on the same axis, from the point *b*; *ac* is a long diagonal of the face coinciding with *ab*.

Because the obtuse angular extremities of a contained rhomboidal dodecahedron, coincide with the obtuse angular extremities of an enveloping trapezohedron: therefore a plane, passing through the obtuse angular extremities *d*, at right angles to the axis *AB*, cuts off one fourth from that axis; and a perpendicular, drawn from the angular extremity *d*, on *A B*, will bisect the semi-axis *Ag*, in *n*; if from the point *b*, where the two diagonals *ac* and *Ad* intersect each other, a perpendicular be drawn on the axis *AB*, the triangle *aAb* will be similar to the triangle *nAd*; therefore as $Ab : Ad :: Aa : An$; but $Ab = \frac{2}{3}\, Ad$ therefore $An = \frac{3}{2}\, An = \frac{1}{2}\, Ag = \frac{1}{6}\, AB$, Therefore, *if a plane &c.*

THEOREM XVI.

The volume of a cube is to the volume of its contained Tetrahedron as 3 : 1.

Because the plane *CDE* (*fig* 36) cuts off a triangular pyramid, and likewise one third[f] of the axis *A B*, which is the altitude of that pyramid; and the remaining part *ab* of the axis is equal to the altitude of the contained tetrahedron; therefore the altitude of the tetrahedron is equal to twice the altitude of the pyramid; but the base *DCE* is common to both; therefore the volume of the tetrahedron is to the volume of the triangular pyramid, as 2 : 1. And because the cube is formed of a tetrahedron and four such pyramids. Therefore, *the volume of a cube will be to the volume of its contained tetrahedron, as* 3 : 1.

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[f] Theo. IX.

THEOREM XVII.

The volume of the Octahedron is to the volume of a Tetrahedron, whose faces are equal to the faces of the octahedron, as 4 : 1.

If an octahedron be divided into two quadrangular pyramids, each of those pyramids may be again divided into two triangular pyramids; which triangular pyramids will have an equal base and altitude with the octahedron; consequently with a tetrahedron having equal faces with the octahedron: therefore the volume of the tetrahedron will be equal, to the volume of the triangular pyramid. But because an octahedron can be divided, into four of the above pyramids; hence the volume of an octahedron is equal, to the volume of four tetrahedrons, whose faces are equal to the faces of the octahedron. Therefore, *an octahedron &c.*

THEOREM XVIII.

The volume of a Tetrahedron is to the volume of its contained octahedron as 2 : 1.

Because a tetrahedron may be divided into four tetrahedrons, and an octahedron, whose faces are equal, see *fig.* 25; and because the volume of a tetrahedron, whose faces are equal to the faces of the octahedron are to that octahedron as 1 : 4[g]. Therefore, *the volume, &c.*

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[g] Theo. XVII.

THEOREM XIX.

The volume of a Cube is to the volume of its inscribed Octahedron, as 6 : 1.

Because a cube is to its contained tetrahedron, as 3 : 1 [h], or 6 : 2; and the tetrahedron is to its contained octahedron as 2 : 1 [i]. And because the octahedron inscribed in the cube, is the same as the octahedron contained in the tetrahedron contained in that cube. Therefore by combining the above ratios it will follow that, *the volume of the cube is to the volume of the inscribed octahedron, as* 6 : 1.

THEOREM XX.

The volume of a Rhomboidal Dodecahedron is to the volume of its contained cube, as 2 : 1.

Because a rhomboidal dodecahedron may be divided by its short diagonals, into six quadrangular pyramids and a cube; and the volume of the six quadrangular pyramids being equal to the volume of the cube.[k] Therefore, *the volume &c.*

THEOREM XXI.

The volume of a Rhomboidal Dodecahedron is to the volume of its contained Octahedron, as 3 : 2.

Because a rhomboidal dodecahedron is equal to its contained octahedron, and two tetrahedrons, whose faces are equal to the faces of the octahedron[l]: and the two tetrahedrons being equal to half the octahedron.[m] Therefore, *the volume &c.*

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[h] Theo. XVI. [i] Theo. XVIII. [k] Cor. VII.

THEOREM XXII.

The volume of a Cuboctahedron is to the volume of its enveloping cube from whence it could be formed, as 5 : 6.

The edge of an octahedron, contained in a tetrahedron, is equal to half the edge of that tetrahedron[n]; and the edge of a tetrahedron, contained in a cube, is equal to the diagonal of that cube[o]: therefore the edge of an octahedron inscribed in a cube, is equal to half the diagonal of the cube: but the edge of a cuboctahedron is likewise equal to half the diagonal of the cube from whence it is formed[p]; consequently, the triangular faces of the cuboctahedron, contained in the cube, is equal to the faces of the octahedron inscribed in the same cube. The triangular pyramids, which must be cut off, from the angular extremities of a cube, to convert it into a cuboctahedron, will have their bases equal to the triangular faces of the contained cuboctahedron; equal to the faces of the inscribed octahedron: consequently, the triangular pyramids will be equal and similar, to the triangular pyramids, into which the contained octahedron might be divided by lines drawn from its angular extremities terminating in its centre[q]. Therefore, the triangular pyramids cut off from a cube, to convert it into a cuboctahedron, is equal to the octahedron inscribed in the same cube; equal to one sixth of the cube[r]. Hence the cuboctahedron is equal to five sixths of the cube from whence it is formed. Therefore, *the volume, &c.*

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[l] Cor. VI. [m] Theo. XVII. [n] Fig. 25. [o] Fig. 26. [p] Theo. V. [q] Theo. VIII. [r] Theo. XIX.

THEOREM XXIII.

The volume of the Cuboctahedron is to the volume of its inscribed Octahedron, as 5 : 1.

Because a cube is to its inscribed octahedron, as 6 : 1[s]; and to its contained cuboctahedron as 6 : 5[t]. And the octahedron inscribed in the cuboctahedron, being identical with the octahedron inscribed in the cube enveloping that cuboctahedron. Therefore by combining the above ratios, it will follow that *the volume, &c.*

THEOREM XXIV.

The volume of the Trapezohedron is to the volume of its contained Rhomboidal Dodecahedron, as 4 : 3.

It was before shewn, page 46, that a trapezohedron may be divided into twenty-four quadrangular pyramids joined base to base in pairs; and that the one half are in an inverted position to the other half; the inverted or interior pyramids forming the contained rhomboidal dodecahedron; and the other pyramids, which may be called the exterior pyramids forming that part of the trapezohedron, which is not common to the solids: and because the two kinds of pyramids have a common base, they will, consequently, be to each other as their altitudes; from whence may be determined the ratios between the volume of the trapezohedron, and its contained rhomboidal dodecahedron.

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[s] Theo. XIX. [t] Theo. XXII.

If a plane pass through one of the long diagonals, of the face of a trapezohedron; and cut the solid at right angles to that face; it will pass through five other long diagonals, making the face of the section a hexagon; and will likewise divide six of the exterior pyramids, and six of the interior pyramids, into two equal parts, passing at the same time through their summits.

To determine the relation, between the volumes of the interior and exterior pyramids; let the face of the above section, of the trapezohedron, be represented by the hexagon *fig* 69. Draw AD, FC, EB and FB; bisect AF in a, and draw ah parallel to FB; through v draw de, parrallel to FB, and join ae; through o draw ib parallel to FB; and join av and hv, ahA will be the section of an exterior pyramid, and Ac its altitude; ahb will be the section of the interior or inverted pyramid, and cv its altitude. The first thing to be determined is the ratio between Ac and cv, the altitudes of the two pyramids.

Ahn and nov being two equal equilateral triangles; the base An will be equal to the base nv; and the perpendiculars hc and oi will bisect these bases, in c and i; making Ac, cn, ni and iv, all equal to each other; therefore the altitude cv will be to the altitude cA, as 3 : 1. And because the two pyramids have a common base, they will be to each other as their altitudes; consequently the volume of the interior pyramid will be to the volume of the exterior pyramid as 3 : 1. But the two pyramids, taken together, are one twelfth part of the trapezohedron; and the interior or inverted pyramid is one twelfth part, of the contained rhomboidal dodecahedron; hence, as these two pyramids taken together are to the interior pyramid; so is the volume of the trapezohedron, to the volume of its contained rhomboidal dodecahedron. Therefore, *the volume, &c.*

THEOREM XXV.

The volume of the Trapezohedron is to the volume of its contained Octahedron, as 2 : 1.

Because the trapezohedron is, to its contained rhomboidal dodecahedron, as 4 : 3 [v]: and the rhomboidal dodecahedron is to its contained octahedron as 3 : 2 [w]: and as the octahedron contained in the rhomboidal dodecahedron is the same as the octahedron contained in the trapezohedron enveloping the rhomboidal dodecahedron; therefore by combining these two ratios, it will be found that *the volume of the trapezohedron will be to the volume of its contained octahedron, as* 4 : 2 or 2 : 1.

THEOREM XXVI.

The volume of a Trapezohedron is the volume of its contained Cube, as 8 : 3.

Because the trapezohedron is to its contained rhomboidal dodecahedron, as 4 : 3 [v] or 8 : 6; and the rhomboidal dodecahedron is to its contained cube, as 2 : 1[y] or 6 : 3; and because the cube, contained in the rhomboidal dodecahedron, is identical with the cube contained in the trapezohedron, enveloping the rhomboidal dodecahedron; therefore by combining these two ratios, it will be found, that *the volume of the trapezohedron is to be the volume of its contained cube, as* 8 : 3.

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[v] Theo XXIV. [w] Theo XXI. [y] Theo. XX.

THEOREM XXVII.

The volume of a Trapezohedron is to the volume of its contained Tetrahedron, as 8 : 1.

Because the trapezohedron is to its contained cube, as 8 : 3[z], and the cube is to its contained tetrahedron, as 3 : 1[a]; and because the tetrahedron contained in the cube is the same, as the tetrahedron contained in the trapezohedron enveloping the cube; therefore by combining these two ratios, it will follow, that *the volume of the trapezohedron is to the volume of its contained tetrahedron, as* 8 : 1.

THEOREM XXVIII.

The volume of an Octahedron is to the volume of its inscribed Rhomboidal Dodecahedron, as 9 : 4.

In the rhomboidal Dodecahedron *fig.* 64, the altitude ae, of the contained octahedron, is to AD, the altitude of the circumscribing octahedron as 2 : 3; and as octahedrons are to each other as the cubes of their altitudes; hence these two octahedrons will be to each other as $2^3 : 3^3 :: 8 : 27$. But the octahedron is to its circumscribed rhomboidal dodecahedron, as 2 : 3[b] or 8 : 12. Therefore, by combining these two ratios, it will be found, that *the rhomboidal dodecahedron is to its circumscribed octahedron, as* 12 : 27 *or* 4 : 9.

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[z] Theo. XXVI. [a] Theo. XVI. [b] Theo. XXI.

THEOREM XXIX.

The volume of an Octahedron is to the volume of its inscribed Trapezohedron, as 27 : 16.

An octahedron containing a rhomboidal dodecahedron, also contains the trapezohedron enveloping the rhomboidal dodecahedron; the obtuse angular extremities of both terminating in the centres of the faces of the octahedron. And as the octahedron is to its inscribed rhomboidal dodecahedron as 9 : 4 [c] or 27 ; 12; and as the rhomboidal dodecahedron is to its enveloping trapezohedron, as 3 : 4 [d] or 12 : 16. Therefore, by combing these ratios, it will follow that *the volume of the octahedron will be to the volume of its inscribed trapezohedron, as* 27 : 16.

THEOREM XXX.

The volume of an Octahedron is to the volume of its inscribed Cube, as 9 : 2.

The angular extremities of a cube inscribed in an octahedron; and the obtuse angular extremities of a trapezohedron inscribed in the same octahedron, both terminate in the centres of the faces of the octahedron: and because an octahedron is to its inscribed trapezohedron as 27 : 16 [e]; and the trapezohedron is to its contained cube, as 8 : 3 [f] or 16 : 6; therefore, by combing these two ratios, it will be found, that *the octahedron is to its inscribed cube, as* 27 : 6 *or* 9 : 2.

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[c] Theo. XXVIII. [d] Theo. XXIV. [e] Theo. XXIX. [f] Theo. XXVI.

THEOREM XXXI.

The volume of an Octahedron is to the volume of its inscribed tetrahedron, as 27 : 2.

Because the cube inscribed in the octahedron, contains the tetrahedron inscribed in the same octahedron; both their angular extremities terminating in the centres of the faces of the octahedron. And because the octahedron is to its inscribed cube, as 9 : 2 [g] or 27 : 6; and the cube is to its contained tetrahedron, as 3 : 1[h] or 6 : 2. Therefore, by combining these ratios, it must necessarily follow, that *the volume of an octahedron is to the volume of its inscribed tetrahedron, as* 27 : 2.

THEOREM XXXII.

The volume of the Rhomboidal Dodecahedron is to the volume of its inscribed trapezohedron, as 16 : 9.

Through *A* (*fig.* 69) draw *tm* parallel to *ab*, and produce *va* and *vh*, to *t* and *m*; the hexagon *ABCDE* and *F* being the section of a trapezohedron, *ab* will be the section of the face of the contained rhomboidal dodecahedron, and *tm* will be the section of the face of a rhomboidal dodecahedron enveloping the trapezohedron; *avh* is the section of one of the quadrangular pyramids, into which the interior rhomboidal dodecahedron may be divided, and *cv* its altitude; *tmv* is the section of a quadrangular pyramid, into which the circumscribing rhomboidal dodecahedron may be divided, and *Av* its altitude. Because the contained rhomboidal dodecahedron is to the circumscribed, as those pyramids are to each other; and these pyramids are to each other as the cubes of their altitudes; and their altitudes which are *cv* and *Av* are to each other as 3 : 4; therefore the contained rhomboidal dodecahedron is to the circumscribed as $3^3 : 4^3$ or 27 : 64. But the contained rhomboidal dodecahedron is to its circumscribed trapezohedron, as 3 : 4[i] or 27 : 36. Therefore, by combining these ratios, it will be found, that *the volume of the rhomboidal dodecahedron is to the volume of its inscribed trapezohedron, as* 64 : 36 or 16 : 9.

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[g] Theo. XXX. [h] Theo. XVI

THEOREM XXXIII.

The volume of a Tetrahedron is to the volume of its contained trapezohedron, as 27 : 8.

Because a trapezohedron inscribed in a tetrahedron is the same, as the trapezohedron inscribed in the octahedron contained in that tetrahedron; and because the tetrahedron is to its contained octahedron, as 2 : 1[k] or 54 : 27; and the octahedron is to its inscribed trapezohedron, as 27 : 16[l]. Therefore, by combining these ratios, it will be found, that *the volume of the tetrahedron is to the volume of its contained trapezohedron, as* 54 : 16 *or* 27 : 8.

THEOREM XXXIV.

The volume of a Tetrahedron is to the volume of its inscribed Rhomboidal Dodecahedron, as 9 : 2.

Because a rhomboidal dodecahedron, inscribed in a tetrahedron, is the same as the rhomboidal dodecahedron inscribed in an octahedron contained in that tetrahedron; and because the tetrahedron is to its contained octahedron, as 2 : 1[m] or 18 : 9; and the octahedron is to its inscribed rhomboidal dodecahedron, as 9 : 4[n]. Therefore, by combining these ratios, it will be found, that *the volume of the tetrahedron is to the volume of its inscribed rhomboidal dodecahedron, as* 18 : 4 *or* 9 : 2.

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[i] Theo. XXIV. [k] Theo. XVIII. [l] Theo. XXIX.

THEOREM XXXV.

The volume of the Tetrahedron is to the volume of its inscribed cube, as 9 : 1.

Because the angular extremities of a cube inscribed in a tetrahedron terminate in the centres of the faces, of the octahedron contained in that tetrahedron; which centres taken alternately are the centres of the faces of the enveloping tetrahedron; and because the inscribed cube is to the octahedron contained in that tetrahedron, as 9 : 2[o]; and the tetrahedron is to its contained octahedron, as 2 : 1[p] or 18 : 9. Therefore, by combining these two ratios, it will be found, that *the volume of the tetrahedron is to its incribed cube, as* 18 : 2 *or* 9 : 1,

THEOREM XXXVI.

The volume of a Cube is to the volume of its inscribed Trapezohedron, as 3 : 1.

Because the angular extremities of an octahedron inscribed in a cube terminate in the centres of its faces; and the acute angular extremities of a trapezohedron inscribed in the same cube, also terminate in the centres of the faces of the cube;

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[m] Theo. XVIII. [n] Theo. XXVIII. [o] Theo. XXX. [p] Theo. XXVIII.

therefore the octahedron inscribed in a cube will be exactly contained, in the trapezohedron inscribed in the same cube; and because an octahedron is to its enveloping trapezohedron, as 1 : 2[q]; and the same octahedron is to its circumscribed cube, as 1 : 6[r]. Therefore, by combining these two ratios, it will follow that *the volumes of the circumscribed cube will be to the volume of its inscribed trapezohedron, as* 6 : 2 *or* 3 : 1.

THEOREM XXXVII.

The volume of a cube is to the volume of its inscribed rhomboidal dodecahedron, as 4 : 1.

Because the angular extremities of an octahedron inscribed in a cube terminate in the centres of its faces; and the acute angular extremities of a rhomboidal dodecahedron inscribed in the same cube likewise terminate in the centres of its faces: therefore an octahedron inscribed in a cube is exactly contained in a rhomboidal dodecahedron inscribed in the same cube; and because the octahedron is to the cube in which it is inscribed, as 1 : 6[s] or 2 : 12; and the octahedron is to the rhomboidal dodecahedron that envelopes it, as 2 : 3[t]. Therefore by combining these two ratios, it will be found, that *the volume of the cube is to the volume of its inscribed rhomboidal dodecahedron, as* 12 : 3 or 4 : 1.

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[q] Theo XXV. [r] Theo XIX. [s] Theo. XIX.
[t] Theo. XXI.

THEOREM XXXVIII.

The volume of a Trapezohedron is to its contained Cuboctahedron, as 27 : 20.

The cube, from which the cuboctahedron contained in a trapezohedron might be made, is called the *supplemental cube;* and the cube, contained in the same trapezohedron, is called the *contained cube,* for the purpose of distinguishing them from each other: the relations between these two cubes must be first determined.

The hexagon *(fig.* 69*)* is the section of a trapezohedron; *AB, BC, CD, &c.* are six long diagonals of the faces of the trapezohedron; but because the long diagonals of a trapezohedron coincide with, and is therefore equal to, the edge of a contained cuboctahedron; hence *EF* is equal to the edge of the contained cuboctahedron; but the edge of a cuboctahedron is equal to half the diagonal of the supplemental cube; therefore the diagonal of the supplemental cube is equal to *AD.*

In the trapezohedron *(fig.* 67,*)* the obtuse angular extremities, *I, b, G* and *h,* being the extremities of the contained cube; therefore a line drawn through the trapezohedron, from *G* to *h* will be one of the diagonals of the contained cube, which is equal to *hg (fig.* 69*)*; but as cubes are to each other as the cubes of their diagonals, and *AD* being equal to the diagonal of the supplemental cube, and *hg* equal to the diagonal of the contained cube; and as *hg* is double *hr,* and *AD* double *Av;* therefore the two diagonals will be to each other as *hr* to *Av,* as 3 : 4; consequently the supplemental cube will be to the contained cube, as $4^3 : 3^3$ or 64 : 27.

But the cuboctahedron is five sixths of the supplemental cube: therefore the cuboctahedron will be to the contained cube, as $\frac{5}{6}$ of $64 : 27 :: \frac{320}{6} : 27 :: \frac{160}{3} : 27 :: 160 : 81 :: 5\times2^5 : 3^4$.

Because the trapezohedron is to its contained cube, as 8 : 3 [u]; and its contained cuboctahedron is to the same cube, as $\frac{160}{9}$: 27 :: $\frac{160}{27}$: 3 : therefore by combining these two ratios, it will follow, that *the volume of the trapezohedron is to the volume of its contained cuboctahedron, as* 8 : $\frac{160}{27}$ :: 1 : $\frac{20}{27}$ :: 27 : 20 :: $3^3 : 2^2 \times 5$.

THEOREM XXXIX.

The volume of the rhomboidal dodecahedron is to the volume of the cuboctahedron, contained in the same trapezohedron with itself, as 81 : 80.

Because the trapezohedron is to its contained rhomboidal dodecahedron, as 4 : 3 [v] or 1 : $\frac{3}{4}$; and to its contained cuboctahedron, as 1 : $\frac{20}{27}$ [w]; therefore, by combining these ratios, the result will be, that *the volume of the rhomboidal dodecahedron is the volume of the cuboctahedron, contained in the same trapezohedron with itself, as* $\frac{3}{4} : \frac{20}{27} :: \frac{81}{108} : \frac{80}{108}$ :: 81 : 80 *or* $3^4 : 2^4 \times 5$.

THEOREM XL.

The volume of the cuboctahedron is to the volume of the octahedron, contained in the same trapezohedron with itself, as 40 : 27.

Because the trapezohedron is to its contained octahedron, as 2 : 1 [x] or 1 : $\frac{1}{2}$, and to its contained cuboctahedron, as 1 : $\frac{20}{27}$ [y].

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[u] Theo. XXVI. [v] Theo. XXIV. [w] Theo. XXXVIII.
[x] Theo. XXV. [y] Theo. XXXVIII.

Therefore, by combining these two ratios, it will follow, that *the volume of the cuboctahedron is to the octahedron, contained in the same trapezohedron with itself, as* $\frac{20}{27} : \frac{1}{2} :: 40 : 27 :: 5 \times 2^3 : 3^3$.

THEOREM XLI.

The volume of the Cuboctahedron is to the volume of the Tetrahedron, contained in the same Trapezohedron, as 160 : 27.

Because the cuboctahedron is to the cube contained in the same trapezohedron with itself, as 160 : 81[z]; and the cube is to its contained tetrahedron, as 3 : 1[a] or 81 : 27; which tetrahedron is identical with the tetrahedron contained in the same trapezohedron with itself. Therefore, by combining these two ratios, the result will be, that *the volume of the cuboctahedron is to the volume of the tetrahedron, contained in the same trapezohedron with itself, as* $160 : 27 :: 5 \times 2^5 : 3^3$.

THEOREM XLII.

The volume of the Cuboctahedron is to the volume of its contained trapezohedron, as 5 : 2.

Because the cube is to its inscribed trapezohedron, as 3 : 1[b] or 6 : 2, and to its contained cuboctahedron, as 6 : 5[c]. Therefore, by combining these ratios, it will be found, that *the volume of the cuboctahedron is to its inscribed trapezohedron, as* 5 : 2.

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[z] Theo. XXXVIII.   [a] Theo. XVI.   [b] Theo. XXXVI.

[c] Theo. XXII.

THEOREM XLIII.

The volume of the Cuboctahedron is to the volume of its inscribed rhomboidal dodecahedron, as 10 : 3.

Because the cube is to its inscribed rhomboidal dodecahedron, as 4 : 1 [d] or 12 : 3, and to its contained cuboctahedron as 6 : 5 [e] or 12 : 10. Therefore, by combining these two ratios together it will follow, that *the volume of the cuboctahedron is to the volume of its inscribed rhomboidal dodecahedron, as* 10 : 3.

THEOREM XLIV.

The volume of the Cuboctahedron is to the volume of its inscribed cube, as 45 : 16.

Because the angular extremities of an inscribed cube terminate in the centres of the triangular faces of the circumscribed cuboctahedron; therefore the edge of the inscribed cube will be equal to the distance, between the centres of two adjoining triangular faces of the cuboctahedron; and the edge of the supplemental cube, or the cube from whence the cuboctahedron is formed will be equal to the diagonal of one of the square faces of the cuboctahedron. The relation between the supplemental cube and the inscribed cube must be first determined.

From *C* (*fig.* 72,) draw two perpendiculars *Co* and *Ca*, on the adjoining triangular faces; these perpendiculars will pass through the centres *b* and *g* of those faces; join *oa*, and from *C* let fall the perpendicular *Cd*, it will bisect *oa* in *d*; and from the centres *b* and *g* let fall the perpendiculars *bc* and *gf*; because the triangles *boc* and *cod* are similar, therefore *bo* : *Co* :: *co* : *do*, and because *Cb* is $\frac{2}{3}$ *Co*, therefore *cd* will be $\frac{2}{3}$ *do*; but *cd* is equal to *df*, therefore *cf* : *oa* :: *cd* : *od* :: 2 : 3; but *oa* is equal to *BE*, equal to the edge of the supplemental cube, and *cf* is equal to the distance between *b* and *g*, equal to the edge of the contained cube; therefore the edge of the contained cube is to the edge of the supplemental cube, as 2 : 3; but cubes are to each other, as the cubes of their edges; therefore the contained cube is to the supplemental cube, as 8 : 27 or $\frac{8}{27}$: 1; and the cuboctahedron being to the supplemental cube as 5 : 6[f] or $\frac{5}{6}$: 1: therefore by combining these two ratios it will be found, that the volume of the cuboctahedron is to the volume of its inscribed cube, as $\frac{5}{6} : \frac{8}{27} :: \frac{135}{162} : \frac{48}{162}$ :: 135 : 48 :: 45 : 16 :: $2^4 : 3^2 \times 5$.

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[d] Theo. XXXVII. [e] Theo. XXII.

THEOREM XLV.

The volume of the Cuboctahedron is to the volume of its inscribed Tetrahedron, as 135 : 16.

The angular extremities of a tetrahedron inscribed in a cuboctahedron terminate in the centres of the triangular faces of the cuboctahedron, and likewise the angular extremities of the inscribed cube; consequently the inscribed cube contains the inscribed tetrahedron; but the inscribed cube is to its circumscribing cuboctahedron as 16 : 45[g] or 48 : 135, and the cube is to its contained tetrahedron, as 3 : 1[h] or 48 : 16: therefore by combining these two ratios, it will follow that *the volume of the cuboctahedron is to the volume of its inscribed tetrahedron, as* 135 : 16 :: $5 \times 3^3 : 2^4$.

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[f] Theo. XXII. [g] Theo. XLIV. [h] Theo. XVI.

THEOREM XLVI.

The volume of a Rhomboidal Dodecahedron is to the volume of its inscribed Cuboctahedron, as 12 : 5.

The angular extremities of a cuboctahedron inscribed in a rhomboidal dodecahedron terminate in the centres of its faces, and bisect the short diagonals; consequently the angular extremities of the cuboctahedron terminate in, and bisect the edges of the contained cube; but the rhomboidal dodecahedron is to its contained cube as 2 : 1[i] or 12 : 6; and the cube is to its contained cuboctahedron, as 6 : 5[k]: therefore, by combining these ratios, it will be found, that *the rhomboidal dodecahedron is to its contained cuboctahedron, as* 12 : 5 :: 3×2^2 : 5.

THEOREM XLVII.

The volume of the Octahedron is to the volume of its contained Cuboctahedron, as 8 : 5.

The centres of the triangular faces of a cuboctahedron contained in an octahedron coincide with the centres of the faces of the octahedron; consequently the cube inscribed in the octahedron is identical with the cube inscribed in the cuboctahedron; and as the octahedron is to its contained cube, as 9 : 2[l] or 72 : 16; and the cuboctahedron is to its contained cube as 45 : 16[m]: therefore by combining these ratios, it will follow that *the volume of the octahedron is to the volume of its contained cuboctahedron, as* 72 : 45 :: 8 : 5 :: 2^3 : 5.

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[i] Theo. XX. [k] Theo. XXII. [l] Theo. XXX. [m] Theo. XLIV.

THEOREM XLVIII.

The volume of the Tetrahedron is to the volume of its contained Cuboctahedron, as 16 : 5.

The cuboctahedron contained in a tetrahedron is identical with the cuboctahedron contained in the octahedron, which is contained in that tetrahedron; and as the tetrahedron is to its contained octahedron, as 2 ; 1[n] or 16 : 8; and the octahedron is to its contained cuboctahedron, as 8 ; 5[o]: therefore, by combing these two ratios, it will follow, that *the volume of the tetrahedron, is to the volume of its contained cuboctahedron, as* 16 : 5 :: 2^4 ; 5.

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[n] Theo. XVIII.

[o] Theo. XLVII.

A TABLE

Shewing the relation between the Volumes of the Natural Solids, when contained in each other.

	to its contained Trapezohedron.	to its contained Rhomboidal Dodecahedron.	to its contained Octahedron.	to its contained Cube.	to its contained Tetrahedron.	to its contained Cuboctahedron
The Trapezohedron is		as 4 : 3	as 2 : 1	as 8 : 3	as 8 : 1	as 27 : 20
The Rhomboidal Dodecahedron is	as 16 : 9		as 3 : 2	as 2 : 1	as 6 : 1	as 12 : 5
The Octahedron is	as 27 : 16	as 9 : 4		as 9 : 2	as 27 : 2	as 8 : 5
The Cube is	as 3 : 1	as 4 : 1	as 6 : 1		as 3 : 1	as 6 : 5
The Tetrahedron is	as 27 : 8	as 9 : 2	as 2 : 1	as 9 : 1		as 16 : 5
The Cuboctahedron is	as 5 : 2	as 10 : 3	as 5 : 1	as 45 : 16	as 135 : 16	

A TABLE

Shewing the ratio between any two similar Natural Solids of different magnitudes, one of them being contained in any one of the Natural Solids, and the other being circumscribed about the same; it is formed by combining the ratios of the foregoing Table.

	Its circumscribed Trapezohedron is to its inscribed, as	Its circumscribed Rhomboidal Dodecahedron is to its inscribed, as	Its circumscribed Octahedron is to its inscribed, as	Its circumscribed Cube is to its inscribed, as	Its circumscribed Tetrahedron is to its inscribed, as	Its circumscribed Cuboctahedron is to its inscribed, as
The Trapezohedron.		$64 : 27 :: 4^3 : 3^3$	$27 : 8 :: 3^3 : 2^3$	$8 : 1 :: 2^3 : 1^3$	$27 : 1 :: 3^3 : 1$	$27 : 8 :: 3^3 : 2^3$
The Rhomboidal Dodecahedron.	$64 : 27 :: 4^3 : 3^3$		$27 : 8 :: 3^3 : 2^3$	$8 : 1 :: 2^3 : 1$	$27 : 1 :: 3^3 : 1$	$8 : 1 :: 2^3 : 1$
The Octahedron.	$27 : 8 :: 3^3 : 2^3$	$27 : 8 :: 3^3 : 2^3$		$27 : 1 :: 3^3 : 1$	$27 : 1 :: 3^3 : 1$	$8 : 1 :: 2^3 : 1$
The Cube.	$8 : 1 :: 2^3 : 1$	$8 : 1 :: 2^3 : 1$	$27 : 1 :: 3^3 : 1$		$27 : 1 :: 3^2 : 1$	$27 : 8 :: 3^3 : 2^3$
The Tetrahedron.	$27 : 1 :: 3^3 : 1$	$27 : 1 :: 3^3 : 1$	$27 : 1 :: 3^3 : 1$	$27 : 1 :: 3^3 : 1$		$27 : 1 :: 3^3 : 1$
The Cuboctahedron.	$27 : 8 :: 3^3 : 2^3$	$8 : 1 :: 2^3 : 1$	$8 : 1 :: 2^3 : 1$	$27 : 8 :: 3^3 : 2^3$	$27 : 1 :: 3^3 : 1$	

DEMONSTRATIONS

OF A VARIETY OF PROPERTIES BELONGING TO THE

ARTIFICIAL SERIES.

THEOREM XLIX.

That the five Platonic bodies are the only solids that can be made of equilateral and equiangular plane figures, without having at the same time re-entering angles.

To demonstrate this, the first thing to be considered is, what variety of equilateral and equi-angular plane figures there are or can be made, from which solids of the above discription might be formed.

The most simple plane figure is the equilateral triangle which has the smallest number of sides a plane figure can possibly be made with, the next in order is the square, the pentagon, the hexagon, &c.

Because a solid angle may be formed of any number of plane angles, not less than three, providing their sum is less than the sum of four right angles; therefore three equilateral triangles placed together will make a solid angle, from whence a *tetrahedron* may be formed; and four equilateral triangles placed together will make a solid angle from whence an *octahedron* may be formed; likewise five equilateral triangles placed together can make a solid angle from whence an *icosahedron* may be formed; because in neither of these cases does the sum of the plane angles amount to four right angles. But six equilateral triangles placed together cannot make a solid angle because their sum equals four right angles.

Three squares placed together will make a solid angle from whence a *Cube* may be formed; but because the angles of a square are right angles, therefore, four of them placed together cannot make a solid angle; consequently a cube is the only solid that can be made entirely of squares.

Three pentagons placed together are capable of making a solid angle from which a *dodecahedron* may be formed, but because the angle of a regular pentagon is greater than a right angle, more than three pentagons cannot be placed together to form a solid angle; therefore the dodecahedron is the only solid that can be made of regular pentagons.

The sum of three angles of a regular hexagon being equal to the sum of four right angles, therefore they are incapable of being placed together to form a solid angle; consequently a solid cannot be formed of hexagons only, and still less of any plane figures with a greater number of sides. Therefore it has been demonstrated that the tetrahedron, the octahedron, the icosahedron, the cube and the dodecahedron are the only solids that can be formed of equal, equilateral and equiangular plane figures, and at the same time have no re-entering angles.

The plane figures hitherto considered are formed entirely of salient, and none of re-entering angles the same may be said of the solids formed of those plane figures.

The French Mathematicians in consequence of making use of re-entering angles have discovered a new species of dodecahedron, agreeing with the dodecahedron already described being formed of twelve equal, equilateral and equiangular pentagons, and yet it has not the least resemblance to the regular dodecahedron at first sight.

In the algebraic calculation, to ascertain the length of the side of a pentagon to be inscribed in a given circle, there arises a quadratic equation which has two possible roots, hence two lines are given of different lengths to form a pentagon within the given circle, consequently there must be two kinds of pentagons, one of these pentagons are formed entirely of salient angles *fig.* 59, and the other of salient and re-entering angles *fig.* 60, which may be called the re-pentagon.

AB (*fig.* 59), is the shortest line, from which that pentagon is formed, and *AB* (*fig.* 60.) is the longest line, from which that re-pentagon is formed, whose sides are *AB*, *BC*, *CD*, *DE*, and *EA*, forming a small pentagon *aboed* in the middle, if the re-pentagon *fig.* 60 be considered as the face of a re-dodecahedron, the small pentagon in the middle will be the face of the ordinary dodecahedron, and the five isosceles triangles around it will be the faces of the pentahedrons, on the adjoining faces of the regular dodecahedron.

The re-dodecahedron (*fig.* 24) is made from the ordinary dodecahedron, by placing pentagonal pyramids on each of its faces, of such a proportion that the inclinations of any one of the pyramids on its base will be equal to the supplement of the inclination of the faces of the ordinary dodecahedron to each other; or the sides of the pyramids must be in the same plane, with the faces of the dodecahedron, adjacent to the face on which the pyramid stands.

If the pyramid *a* of the re-dodecahedron, *fig.* 24, be removed, one of the faces of the re-dodecahedron will become visible, whose sides will be *dn*, *nc*, *ce*, *eb*, and *bd*; which is similar to the re-pentagon, *fig.* 60.

THEOREM L.

That the triangular faces of a Pentahedron have each of the angles at the base double the third angle.

Let the pentagon *abced (fig.* 60,*)* be the face of a contained dodecahedron, *Ada* the face of a pentahedron, on one of the adjoining faces of the dodecahedron, *Aa* and *ab* will form one continued straight line. Because the angle *dab* of a regular pentagon is 108°. therefore *Aad*=72°=*Ada;* and therefore *dAa*=36° equals half 72°, equals half the angle at the base. *Therefore the triangular faces, &c.*

THEOREM LI.

That the Pentagon drawn on the triangular face of a Pentahedron is equilateral and equiangular.

The method of drawing this pentagon has been given in page 63.

The triangular face of a pentahedron is an isosceles triangle having the angles of the base each double the third angle; therefore the angle *BCA (fig.* 54*)* and *CBA* are each double *CAB*. *Aa*=*aB*, and *ba* is perpendicular to *AB* therefore *bA*=*bB*: because *cA*=*cC*, and *cd* is perpendicular to *AC* therefore *dA*=*dC:* and because the side *AB*=*AC* and the angle *BAC* common to both the triangles *bAB* and *dAC,* therefore they will be equal and similar to each other, and to the triangle *CrB*, the angles of whose base are each equal to the angle *CAB*. But *CrB*=*brd,* therefore *nbr*=*brd*=*rde,* therefore *Brd*=*Bdr*, and because *Ane*=*Aee,* the two triangles *nAe* and *rBd* are similar, and because Bd equals Ae they are equal: hence $enb=ned=edr$, therefore the pentagon is equiangular.

Because the triangles nAe, dBr, and rCb are equal and similar, $ne=dr=rb$; and because $Ad=dC$ and $eA=rC$, therefore $de=dr$: likewise $nb=br$. Therefore the pentagon $nedrb$ is equilateral, which was to be demonstrated.

Three solids having been described, namely, the rhombohedron, the rhomboidal dodecahedron and the triacontahedron, each of which are formed of equal and similar rhombs; and likewise three solids, namely, the trapezoidal dodecahedron, the trapezhedron, and the hexacontahedron, each of which are formed of equal and similar trapezoidal faces; some readers may be curious to know, if it were possible to make other solids of any other number of equal and similar rhombic, or trapezoidal faces, than those already employed: to gratify such readers the following theorems are introduced.

THEOREM LII.

The Rhomboidal Dodecahedron and Triacontahedron are the only solids that can be formed of equal and similar rhombs, grouped together in such a manner, that their obtuse plane angles meet to form the obtuse solid angles; and their acute plane angles meet to form the acute solid angles.

Because a solid angle cannot be made of more than three equal obtuse plane angles, each of which plane angles must be less than 120°, therefore the acute plane angles of each of the rhombs must be greater than 60°, but more than five equal plane angles each greater than 60° cannot make a solid angle, therefore five rhombs is the greatest number that can be united together by their acute plane angles to form a solid angle:

hence the triacontahedron is formed of the greatest number of equal and similar rhombs with which a solid can be made. Four rhombs may be united together by their acute angles, from whence may be formed a rhomboidal dodecahedron; which solid ranks next to to the triacontahedron. Three rhombs may be united by their acute plane angles, from whence an acute rhombohedron might be formed; but this solid does not answer the conditions stated, which are, that the angular extremities should be each formed of equal plane angles, whereas one set of its angular extremities are formed of two obtuse plane angles and one acute. If three rhombs were united by their obtuse plane angles, they would form the summit of an obtuse rhombohedron, but this solid, like the acute, does not answer the conditions, agreeing only with one half; the acute rhombohedron having one set of its angular extremities formed by the meeting of equal acute plane angles, and the obtuse rhombohedron having one set of its angular extremities formed by the meeting of equal obtuse plane angles, so that taken together, they answer the conditions, though neither of them separately *Therefore, the rhomboidal dodecahedron, &c.*

THEOREM LIII.

There is no solid that can be formed of equal and similar Trapezoidal faces grouped together like the faces of the Trapezohedron with any other number of faces than either 12, 24, *or* 60.

The angles of the faces of a trapezohedron are of three kinds, one obtuse, one acute and two equal and opposite, and as more than three equal obtuse plane angles cannot form a solid angle, each of which must be less than 120°, and because four of the equal and opposite angles meet together to form the intermediary solid angle, each of which must consequently be less than 90°; therefore the sum of the obtuse and the two equal and opposite plane angles must be less than 300°; hence the acute plane angles must be each greater than 60°, therefore, six of the above trapezoidal faces cannot be united together by their acute plane angles to form a solid angle, consequently the hexacontahedron is formed of the greatest possible number of equal and similar trapezoidal faces, grouped together in a similar manner, as the faces of the trapezohedron. Because the acute angular extremities of a trapezohedron are formed by the meeting of the acute angles of four trapezoidal faces, therefore it is next in order to the hexacontahedron. And because the acute angular extremities of the trapezoidal dodecahedron is formed by the meeting of the acute angles of three trapezoidal faces, therefore it is next in order to the trapezohedron. And as a solid angle cannot be formed of less than three plane angles, therefore the trapezoidal dodecahedron is formed of the least possible number of equal and similar trapezoidal faces. But the hexacontahedron, the trapezohedron and the trapezoidal dodecahedron may each have their axis either elongated or shortened without altering the general form of their faces, hence an indefinite variety of solids may be formed with either 12, 24, or 60 equal and similar trapezoidal faces, but not with any other number. *Therefore, there is no solid, &c.*

APPENDIX.

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APPENDIX.

PROBLEM I.

Required, in the regular Dodecahedron, the angles of inclination of the faces and edges to the axis passing through one of the angles; and also the mutual inclination of two adjoining faces.

Let ABCDE (*fig.* 59.) be one of the faces of the dodecahedron: let AB, one of the sides, be $= e$; AE, an oblique diagonal, $= b$. Then e is to b, as the smaller portion of a line divided into extreme and mean ratio is to the larger; that is, as 1 to $\frac{1}{2}(1+\sqrt{5})$. Let $\frac{1}{2}(1+\sqrt{5}) = m$: then will $b = m\,e$.

Let GHIJKLM (*fig.* 50.) be the section of a dodecahedron passing through the centre C and two opposite edges, GH and LK; GM, ML, and HI, IK, being perpendiculars on the lateral faces from the angles G, H, L, K; and GK, HL, being axes drawn through the same angular points.

Let NPQR be the section of the inscribed, or generating cube: the side of which, NP, will be equal to the oblique diagonal of the pentagonal face, or b. Draw NH, which will be the section of the equilateral triangle formed by the oblique diagonals of the three faces which compose the solid angle G, and which subtend that angle. NH will be the perpendicular from the vertex H, of that triangle, to the base: it will therefore be to the base b, as 1 to $\frac{1}{2}\sqrt{3}$; or, calling it p, $p = \frac{1}{2}b\sqrt{3} = \frac{1}{2}me\sqrt{3}$. The axis GK will be perpendicular to the triangle, and therefore perpendicular to NH. Draw MI through the centre, and parallel to GH.

Let the angle HGX $= \varepsilon$

———— NGX $= \phi$

———— GMI $= \eta$

Sin. HGX $= \frac{HX}{GH}$, or, HX being $= \frac{2}{3}$ NH $= \frac{2}{3}p$, and GH $= e$,

$$\text{Sin. } \varepsilon = \frac{2}{3}\,\frac{p}{e} = \frac{m}{\sqrt{3}}:$$

$$\text{Tang. } \varepsilon = \frac{\text{Sin. } \varepsilon}{\sqrt{1 - \text{Sin.}^2\, \varepsilon}} = \frac{m}{\sqrt{3}}\sqrt{\frac{3}{3-m^2}} = m^2$$

NX being $= \frac{1}{3}$ NH $= \frac{1}{2}$ XH,

$$\therefore \text{Tang. } \phi = \frac{1}{2}\,\text{Tang. } \varepsilon = \frac{1}{2}\, m^2$$

HGM and GMI being supplementary angles,

$$\text{Tang. } \eta = -\text{Tang. } (\varepsilon + \phi) = -\frac{m + \frac{1}{2}m^2}{1 - \frac{1}{4}m^4} = m.$$

Hence, $\text{Tang. } \varepsilon = m^2 = \frac{1}{2}(3 + \sqrt{5}) = 2.618034$:

$\text{Tang. } \phi = \frac{1}{2}m^2 = \frac{1}{4}(3 + \sqrt{5}) = 1.309017$:

$\text{Tang. } \eta = m = \frac{1}{2}(1 + \sqrt{5}) = 1.618034$:

$$\text{The Angles} \left\{ \begin{array}{l} \varepsilon = 69^\circ\ 5'\ 41''\ \tfrac{1}{2}. \\ \phi = 52^\circ\ 3'\ 721''\ \tfrac{1}{2} \\ \eta = 58^\circ\ 16'\ 57''. \end{array} \right.$$

Therefore the angles of inclination of the axis with the edge $(= \varepsilon) = 69^\circ\ 5'\ 41''\ \frac{1}{2}$; with the face $(= \phi) = 52^\circ\ 37'\ 21''\ \frac{1}{2}$; and the mutual inclination of the faces $(= 2\eta) = 116^\circ\ 33'\ 54''$

PROBLEM II.

To find in the regular Icosahedron, the angles of inclination of the axis with the faces and edges; and the mutual inclination of two adjacent faces.

Let B *b* *f* B′ *b*′ F (*fig.* 49.) be a section of a dodecahedron through the centre C and two opposite edges B *b* B′ *b*, and L *l* *g* L′ *l*′ G be a section of an icosahedron, having the same centre as the dodecahedron, and having the axes, as L L′ and *l* *l*′, passing through the angular points, perpendicular to the faces of the dodecahedron, as B F, F *b*′, *b* *f*, B *f*: and such that the edge L*l* shall in its middle point F, coincide with the edge of the dodecahedron formed by the junction of the two faces of which the sections are F B and F *b*′. The edges of the two figures thus reciprocally bisecting each other at right angles; the edges common to the planes of the faces of the icosahedron, of which the sections are L G, *l*′ G, *l* *g*, *l*′ *g*, will therefore coincide with the points G and *g*, bisecting the edges B *b*, B′ *b* of the dodecahedron.

L C being perpendicular to B F, and B C to L G, the angle F L C, which is the angle between the axis and edge of the icosahedron, is equal to B C F, or half the angle of the mutual inclination of the faces of the dodecahedron ($=\eta$) of which the tangent was found, by the former problem, to be $= m$, or $\frac{1}{2}(1+\sqrt{5})$.

Also the angle G L C, or that between the axis and face of the icosahedron, is equal to C B F, both being the complement of L C B; and therefore, by the last problem, its tangent $=\frac{1}{2}m^2$.

Lastly, the angle L G C, or the mutual inclination of the faces of the icosahedron = G B C, = s, of which the tangent is m^2. Therefore, in the icosahedron, the angle between the axis and the edge = 58° 16′ 57″; between the axis and face = 52° 37′ 21″ ½; and the mutual inclination of the faces = $2s$ = 138° 11′ 23″.

Corollary. The angle L C B between the axes of the two figures, is the complement of C L G or C B λ; and is therefore = 37° 22′ 38″½.

The same results may be obtained quite independently of the dodecahedron, by drawing, in the section as before described, the line L g, which will be the section of the base of a pentagonal pyramid, having the angle at l for its summit. L g will be the perpendicular drawn from the angle L of the pentagon to the opposite side; and it will be divided by the perpendicular axis lc into two portions, L V, V g. Let the side E D of the pentagonal base, (*fig.* 59,) be $= e$, A E $= b$, A P, (perpendicular to, and bisecting E D,) $= p$. A C $= w$, C P $= v$.

Draw C Z perpendicular to A E; $AZ = \frac{1}{2} b = \frac{1}{2} e m$. $AP = \sqrt{AE^2 - EP^2}$ or $p = \sqrt{b^2 - \frac{1}{4} e^2} = \sqrt{e^2 m^2 - \frac{1}{4} e^2} = e \sqrt{m^2 - \frac{1}{4}} = e \frac{1}{2} \sqrt{4 m^2 - 1} = e \frac{1}{2} \sqrt{m^3 \sqrt{5}}$. $AP :: AE :: AZ : AC = \frac{AE.\ AZ}{AP} = \frac{b^2}{2p} = \frac{e^2 m^2}{e \sqrt{m^3 \sqrt{5}}} = e \sqrt{\frac{m}{\sqrt{5}}} = w$.

$AC\ (= AP - AC) : CP :: w : p - w\ (= v) :: e \sqrt{\frac{m}{\sqrt{5}}} : e \frac{1}{2} \sqrt{m^3 \sqrt{5}} - e \sqrt{\frac{m}{\sqrt{5}}} :: 1 : \frac{1}{2} m$ and $v = m w$.

Let the angle L *l* *c* (*fig.* 51) $= \eta$; L*l* $= e$; LV $= e\sqrt{\frac{m}{\sqrt{5}}}$;

$$Vg = \tfrac{1}{2}\, m\, LV = \tfrac{1}{2}\, e\, m \sqrt{\frac{m}{\sqrt{5}}} = \tfrac{1}{2}\, e \sqrt{\frac{m^2}{\sqrt{5}}};$$

$$\text{Sin. } \eta = \frac{LV}{L\,l} = \sqrt{\frac{m}{\sqrt{5}}};$$

$$\text{Tang. } \eta = \frac{\text{Sin. } \eta}{\sqrt{1 - \text{Sin.}^2\, \eta}} = \sqrt{\frac{m}{\sqrt{5}}} \sqrt{\frac{1}{1 - \frac{m}{\sqrt{5}}}} = m.$$

$$\text{Tang. } \phi = \tfrac{1}{2}\, m; \qquad \text{Tang. } \eta = \tfrac{1}{2}\, m^2;$$

$$\text{Tang. } \varepsilon = -\text{tang.}(\eta + \phi) = -\frac{\text{tang. } \eta + \text{tang. } \phi}{1 - \text{tang. } \eta . \text{ tang. } \phi}$$

$$= -\frac{m + \frac{1}{2} m^2}{1 - \frac{1}{2} m^3} = m^2$$

PROBLEM III.

To find the ratio between the volumes of the Triacontahedron, and its inscribed dodecahedron and icosahedron.

Let the Octagon L *l* *b′* B′ L′ *l′* *b* B (*fig.* 49) be the section of the rhomboidal triacontahedron passing through the centre, C, parallel to one of the faces: and let B *b* *f* B′ *b′* F be the section of the inscribed dodecahedron, and L *l* *g* L′ *l′* G, that of the inscribed icosahedron; L L′, *l* *l′*, being two of the longer axes passing through the summits of the five acute angles of five adjacent faces, and B′B. *b* *b′*, the shorter diameters passing through the summits of the three obtuse angles of three adjacent faces. The former being the axes of the icosahedron, the latter the axes of the dodecahedron. L *l*, L′ *l′*, will be the longer diagonals; B *b*, B *b′*, the shorter diagonals of the rhomboidal faces. L B, L′ B′ will be two opposite edges of the triacontahedron, as also *l* *b*, l′ *b′*; which will be bisected in F, *f*, G, *g* respectively.

These points will be equally distant from the centre, or $CF = CG = Cf = Cg$.

The solid pentahedral angles having L, L', l, l', for their summits, when cut by a base having the short diagonals of the faces for their sides, compose pentagonal pyramids; BF, Fb', bf, fB' being the sections of the bases, and the perpendiculars, in these bases, from the angular points B, b, B', b' to their opposite sides.

The inscribed dodecahedron may in like manner be divided into twelve pentagonal pyramids, having each of the faces for the base of each pyramid, and their common vertex being at the centre of the solid. The former pyramids will be to the latter as the perpendicular $L\lambda$ to the perpendicular $C\lambda$; or the whole of these, composing the triacontahedron, will be to the latter, or the inscribed dodecahedron, as CL to $C\lambda$.

In like manner the solid trihedral angles B, b, B', b', compose pyramids by sections through the long diagonals of the faces of the triacontahedron, of the bases of which LG, lg, $L'G'$, $l'g'$ are the perpendiculars: and dividing the inscribed icosahedron into twenty pyramids, having one of the faces as the base of each, and making the centre their common vertex: the triacontahedron will be to the icosahedron as the united perpendiculars $C\beta$ and βB, or CB, is to $C\beta$.

Let the volume of the triacontrahedron be $= T$: that of the inscribed dodecahedron $= D$, and that of the inscribed icosahedron $= I$.

Let $CB = R$, $C\beta = r'$; $CL = R'$ and $C\lambda = r$.

As before let the angle CBG$=\varepsilon$; CLF$=\eta$: In the triangles CBG and CLF the angles LFC and BCG being right angles and CF = CG; CB : CL (:: R : R′) :: Cosec. CBG : Cosec. CLF :: Sin. η : Sin. ε :: $\sqrt{\frac{m}{\sqrt{5}}} : \frac{m}{\sqrt{3}} :: \sqrt{3} : \sqrt{m\sqrt{5}}$;

CL : Cβ :: Radius : Sin. CLβ = Sin. CBF=Sin. ϕ.

$$\text{Tang. } \phi = \frac{1}{2}m^2 ; \quad \text{Sin. } \phi = \frac{\text{tang. } \phi}{\sqrt{1 + \text{tang}^2.\ \phi}} = \frac{m}{2\sqrt{1+\frac{1}{4}m^4}} =$$

$$\sqrt{\frac{m^3}{3\sqrt{5}}} \text{ or } R' : r :: 1 : \sqrt{\frac{m^3}{3\sqrt{5}}}$$

Therefore R : r :: T : I :: 3 : m^2 :: 6 : 3+ $\sqrt{5}$.

The similarity of the triangles LCβ and BCλ gives us

Cλ : Cβ :: CB : CL :: r' : r :: R′ : R :: $\sqrt{m\sqrt{5}} : \sqrt{3}$;

Therefore R′ : r' :: T : D :: $\sqrt{5} : m :: 2\sqrt{5} : 1+\sqrt{5}$.

D : I :: 3 : $m\sqrt{5}$ :: 6 : 5+$\sqrt{5}$.

Lλ : Cλ :: $\sqrt{5}-1 : \sqrt{5}+1$ :: 16 : 3+$\sqrt{5}$.

Bβ : Lλ :: 3 — 5 : 3+$\sqrt{5}$ :: Lλ^2 : Cλ^2.

Corollary 1. The tangent of the angle CLB, between the long axis and edge of the triacontahedron = 2; for it is to the tangent FLC (=m) as Bλ : λF :: m : 2. The angle is therefore = 63° 26′ 6″.

2. The tangent of the angle CBL, between the short axis and the edge of the triacontahedron = twice the tangent CBG, (for Lβ=2βG) = 2m^2. The angle is therefore = 79° 11′ 16′.

PROBLEM IV.

To find the ratio between the Hexacontahedron and its inscribed Triacontahedron.

The section of the Hexacontahedron through the centre and the shorter diagonal of a face, presents a regular decagon, of which let XY, XY′ (*fig.* 52) be two of the sides, C being the centre. Let CX = R. The bases of each quadrilateral pyramid formed by four long diameters of the four adjacent sides, will coincide with the faces of the inscribed triacontahedron. The section, for instance, of the base of that pyramid whose apex is at X, will be BB′, and the perpendicular from the apex of the pyramid to the base, will be XP = v.

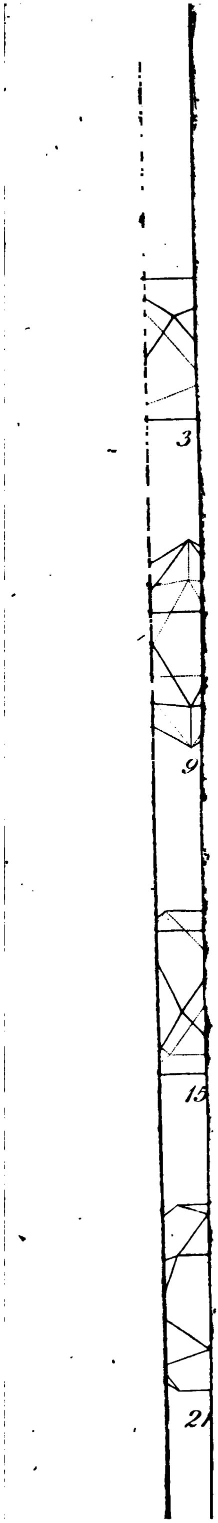
Let the inscribed triacontahedron be divided into thirty pyramids, having its faces for their respective bases, and their common vertex being at the centre. Each of these will be to each of the former as the perpendiculars CP to XP, since they have the same base common to both. The whole of the pyramids therefore, that is the whole hexacontahedron (= H) will be to the latter, or the triacontahedron (= T) as CX to CP :: R : R—v.

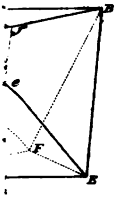
$$XY = R\frac{1}{m};\ XZ = R\frac{1}{2m};\ XP = \frac{(XZ)^2}{R},\ \text{or}\ v = R\frac{1}{4m^2};$$

$$H : T :: R : R - v :: 4m^2 : 4m^2 - 1 :: 2m : \sqrt{5} :: 1 + \sqrt{5} : \sqrt{5}.$$

Corollary. $H : I :: 6 : m\sqrt{5} :: 12 : 5 + \sqrt{5};\ H : D :: 2 : 1.$

FINIS.

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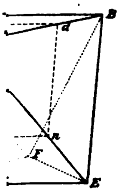
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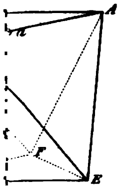
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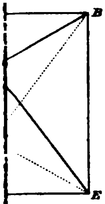
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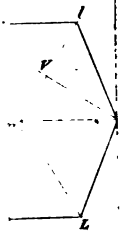

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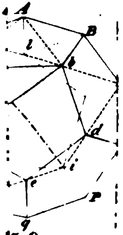

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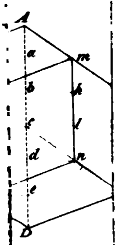

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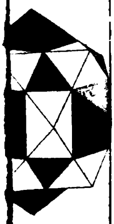

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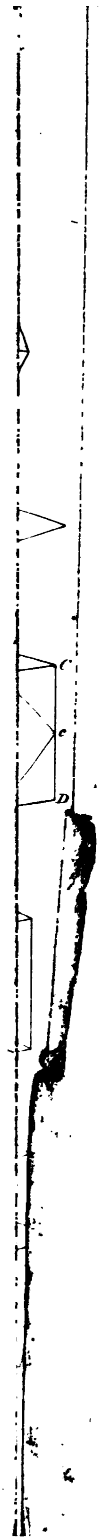
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—— *ib. last line, for* sixt, iav hngy *read* sixty having.
—— 7, *line* 4 *from the bottom, for Ata read Aa*
—— 10, *line* 20, *for* 84 *read* 48
—— 11, *line* 1, *for* 92 *read* 80
—— 25, *line* 13, *for ud read uD*
—— 31, *lines* 7 *and* 10, *for* concide *read* coincide
—— 34, *line* 1, *for* octahedron *read* cube
—— 36, *line* 11, *for* 46″ *read* 47″
—— *ib. line* 4 *from the bottom, for* 11 *read* $\sqrt{11}$
—— 38, *line* 3, *for* 30 *read* 6
—— 39, *line* 10, *for* 70 *read* 71
—— 53, *last line, for* are *read* is
—— 54, *line* 11, *for* 32, *read* $\sqrt{32}$
—— 64, *line* 2, *for* dodeuahedron *read* dodecahedron
—— *ib.* 4, *from the bottom, for* of *read* if
—— 69, *line* 6, *for* 96 *read* 100
—— 90, *line* 5, *for eDc read eDC*
—— 107, *line* 14, *for ahb read ahv*
—— 108, *line* 13, *for* is the *read* is to the
—— 111, *line* 17, *for ab read ah*
—— 127. *last line, for Aee read Aen*
—— 134, *line* 5 *from the bottom, for* $\phi = 52^\circ\ 3'\ 721'\ \frac{1}{2}$ *read* $\phi = 52^\circ\ 37'\ 21''\ \frac{1}{2}$.
—— 136, *last line, for* $v = m\ w$ *read* $v = \frac{1}{2}\ m\ w$.
—— 137, *line* 5, *for* $\frac{1}{2}\,m$; Tang. η *read* $\frac{1}{2}\,m \times$ Tang. η
—— 139, *line* 14, *for* 3—5, *read* $3-\sqrt{5}$.